2.016 Hydrodynamics

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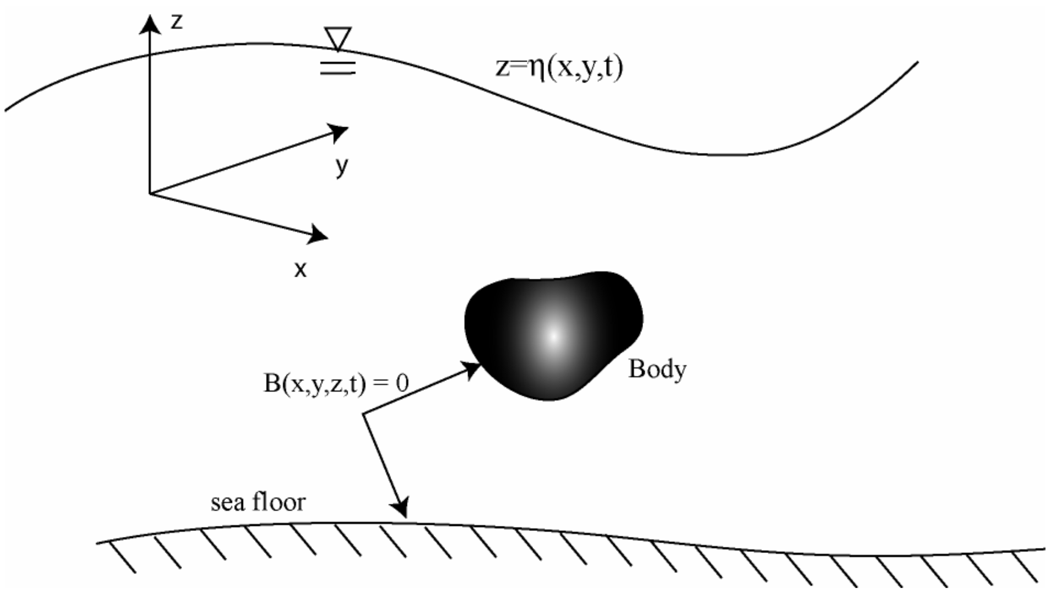
Free Surface Water Waves

I. Problem setup



1. Free surface water wave problem.

In order to determine an exact equation for the problem of free surface gravity waves we will assume potential theory (ideal flow) and ignore the effects of viscosity. Waves in the ocean are not typically uni-directional, but often approach structures from many directions. This complicates the problem of free surface wave analysis, but can be overcome through a series of assumptions.

To setup the exact solution to the free surface gravity wave problem we first specify our unknowns:

- Velocity Field: $\vec{V}(x,y,z,t) = \nabla\phi(x,y,z,t)$
- Free surface elevation: $\eta(x,y,t)$
- Pressure field: $p(x,y,z,t)$

Next we need to set up the equations and conditions that govern the problem:

- Continuity (Conservation of Mass):
  - $\nabla^2\phi = 0$ for $z < \eta$ (Laplace's Equation) (7.1)
- Bernoulli's Equation (given some ϕ):
  - $\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + \frac{p-p_a}{\rho} + gz = 0$ for $z < \eta$ (7.2)
- No disturbance far away:
  - $\frac{\partial\phi}{\partial t}$, $\nabla\phi \to 0$ and $p = p_a - \rho g z$ (7.3)

Finally we need to dictate the boundary conditions at the free surface, seafloor and on any body in the water:

(1) Pressure is constant across the free surface interface: $p = p_{atm}$ **on** $z = \eta$**.**

$$p = -\rho\left\{\frac{\partial\phi}{\partial t} - \frac{1}{2}V^2 - gz\right\} + c(t) = p_{atm}. \tag{7.4}$$

Choosing a suitable integration constant, $c(t) = p_{atm}$, the boundary condition on $z = \eta$ becomes

$$\rho\{\frac{\partial\phi}{\partial t} + \frac{1}{2}V^2 + g\eta\} = 0. \tag{7.5}$$

(2) Once a particle is on the free surface, it remains there always. Similarly, the normal velocity of a particle on the surface follows the normal velocity of the surface itself.

$$z_p = \eta(x_p, t)$$

$$z_p + \delta z_p = \eta(x_p + \delta x_p, t + \delta t) = \eta(x_p, t) + \frac{\partial\eta}{\partial x}\delta x_p + \frac{\partial\eta}{\partial t}\delta t \tag{7.6}$$

On the surface, where $z_p = \eta$, we can reduce the above equation to

$$\delta z_p = \frac{\partial\eta}{\partial x}u\,\delta t + \frac{\partial\eta}{\partial t}\delta t \tag{7.7}$$

and substitute $\delta z_p = w\,\delta t$ and $\delta x_p = u\,\delta t$ to show that the normal velocity follows the particle:

$$w = u\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} \; on \; z = \eta. \tag{7.8}$$

(3) On an impervious body boundary $B(x, y, z, t) = 0$**.** Velocity of the fluid normal to the body must be equal to the body velocity in that direction:

$$\vec{v} \cdot \hat{n} = \quad \phi \cdot \hat{n} = \frac{\partial \phi}{\partial n} = \vec{U}(\vec{x}, t) \cdot \hat{n}(\vec{x}, t) = U_n \;\; on \;\; B = 0. \tag{7.9}$$

Alternately a particle P on B remains on B always; ie. B is a material surface.

For example: if P is on B at some time $t = t_o$ such that

$$B(\vec{x}, t_o) = 0, \; then \;\; B(\vec{x}, t_o) = 0 \;\; for \;\; all \;\; t, \tag{7.10}$$

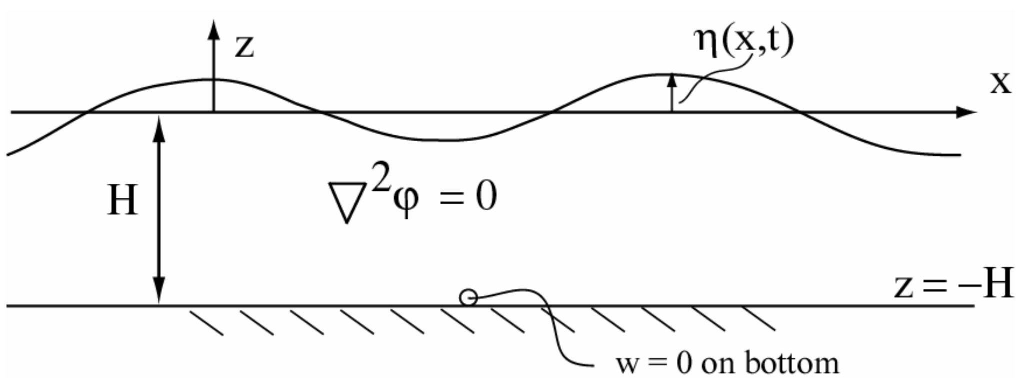
so that if we were to follow P then $B = 0$ always. Therefore:

$$\frac{DB}{Dt} = \frac{\partial B}{\partial t} + (\nabla \phi \cdot \nabla) B = 0 \;\; on \;\; B = 0. \tag{7.11}$$

Take for example a flat bottom at $z = -H$:

$$\partial \phi / \partial z = 0 \;\; \text{on} \;\; z = -H \tag{7.12}$$

II. Linear Waves



2. Linearized Wave Problem.

To simplify the complex problem of ocean waves we will consider only small amplitude waves (such that the slope of the free surface is small). This means that the wave amplitude is much smaller than the wavelength of the waves.

As a general rule of thumb for wave height to wavelength ratios (h/λ, where h is twice the wave amplitude) less than 1/7 we can linearize the free surface boundary conditions. Non-dimensional variables can be used to assess which terms can be dropped.

Non-dimensional variables:

$$\eta = a\,\eta^* \qquad \omega\, t = t^*$$
$$u = a\omega\, u^* \qquad x = \lambda\, x^*$$
$$w = a\omega\, w^* \quad \phi = a\omega\lambda\, \phi^*$$

Looking back to equation (7.5) we can evaluate the relative magnitude of each term by plugging in the non-dimensional versions of each variable. For example:

$$d\phi = a\omega\lambda\, d\phi^*$$
$$dt = 1/\omega\, dt^*$$
$$dx = \lambda\, dx^*$$

Compare $\frac{\partial\phi}{\partial t}$ and $V^2 \sim (\frac{\partial\phi}{\partial x})^2$ to determine which terms can be dropped from equation (7.5):

$$\frac{(\frac{\partial\phi}{\partial x})^2}{\frac{\partial\phi}{\partial t}} = \frac{a^2\omega^2}{a\omega^2\lambda}\,\frac{(\frac{\partial\phi^*}{\partial x^*})}{\frac{\partial\phi^*}{\partial t^*}} = \frac{a}{\lambda}\,\frac{(\frac{\partial\phi^*}{\partial x^*})}{\frac{\partial\phi^*}{\partial t^*}}\,. \tag{7.13}$$

Here we see that if $h/\lambda << 1/7$, then

$$(\frac{\partial\phi}{\partial x})^2 << \frac{\partial\phi}{\partial t}, \tag{7.14}$$

and we can drop the smaller term resulting in linearized boundary conditions

$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial z}\,,$$
$$\frac{\partial\phi}{\partial t} + g\eta = 0\,,$$

which are applicable on $z = \eta$.

Throughout this discussion we have assumed that the wave height is small compared to the wavelength. Along these lines we can also see why η is also quite small. If we expand $\phi(x,z,t)$ about $z=0$ using Taylor series:

$$\phi(x,z=\eta,t)=\phi(x,0,t)+\frac{\partial\phi}{\partial z}\eta+\ldots, \tag{7.15}$$

it can be readily shown that, the second term, $\frac{\partial\phi}{\partial z}\eta$ and the subsequent higher order terms are very small and can be ignored in our linear equations allowing us to rewrite the boundary conditions at $z=\eta$ as boundary conditions on $z=0$. 0.2in

$$\frac{\partial^2\phi}{\partial t^2}+g\,\frac{\partial\phi}{\partial z}=0 \tag{7.16}$$

$$\eta=-\frac{1}{g}\,\frac{\partial\phi}{\partial t} \tag{7.17}$$

on $z=0$.

A Solution to the Linear Wave Problem

The complete boundary value wave problem consists of the differential equation, specifically Laplace's Equation,

$$\nabla^2\phi(x,z,t)=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{7.18}$$

with the following boundary conditions:

1) Bottom Boundary Condition:

$$\frac{\partial\phi}{\partial z}=0 \; on \; z=-H \tag{7.19}$$

2) Free Surface Dynamic Boundary Condition (FSDBC):

$$\eta=\frac{1}{g}\left(\frac{\partial\phi}{\partial t}\right) \; on \; z=0. \tag{7.20}$$

3) Free Surface Kinematic Boundary Condition (FSKBC):

$$\frac{\partial^2 \phi}{\partial t^2} + g\,\frac{\partial \phi}{\partial z} = 0 \tag{7.21}$$

Through separation of variables we can solve Laplace's Equation for η and ϕ.

$$\eta(x,t) = a\,\cos(kx - \omega t + \psi) \tag{7.22}$$

$$\phi(x,z,t) = -\frac{a\omega}{k}\,f(z)\sin(kx - \omega t + \psi) \tag{7.23}$$

$$u(x,z,t) = a\omega\,f(z)\cos(kx - \omega t + \psi) \tag{7.24}$$

$$w(x,z,t) = -a\omega f_1(z)\sin(kx - \omega t + \psi) \tag{7.25}$$

$$f(z) = \frac{\cosh[k(z+H)]}{\sinh(kH)} \tag{7.26}$$

$$f_1(z) = \frac{\sinh[k(z+H)]}{\sinh(kH)} \tag{7.27}$$

$$\omega^2 = gk\,\tanh(kH) \Rightarrow \textit{dispersion relation} \tag{7.28}$$

where a, ω, k, ψ are integration constants with physical interpretation: a is the wave amplitude, ω the wave frequency, k is the wavenumber ($k = 2\pi/\lambda$), and ψ simply adds a phase shift.

III. Dispersion Relation

The dispersion relationship uniquely relates the wave frequency and wave number given the depth of the water. The chosen potential function, ϕ, MUST satisfy the free surface boundary conditions (equation (7.20) and (7.21)) such that plugging ϕ in to the FSKBC (eq. (7.21)) we get:

$$-\omega^2 \cosh kH + gk\sinh kH = 0 \tag{7.29}$$

Resulting in the Dispersion relationship:

$$\omega^2 = gk\tanh kH; \tag{7.30}$$

For deep water where $H \to \infty$ $\tanh kH \to 1$ so that the dispersion relationship in deep water is:

$$\omega^2 = gk \tag{7.31}$$

In general, $k \uparrow$ as $\omega \uparrow$ or $\lambda \uparrow$ as $T \uparrow$.

Phase and Group Speed

Phase speed, C_p, of a wave (velocity that a wave crest is traveling at) is found in general using:

$$\frac{\lambda}{T} = C_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh kH}\;. \tag{7.32}$$

This simplifies for the case of deep water such that

$$C_p = \sqrt{\frac{g}{k}} \tag{7.33}$$

Solution to the dispersion relationship in general form can be found graphically.

IV. Pressure under a Wave

The pressure under a wave can be found using the unsteady form of Bernoulli's equation and the wave potential, $\phi(x,z,t)$:

$$p = \underbrace{-\underbrace{\rho\frac{\partial\phi}{\partial t}}_{\substack{\text{unsteady}\\ \text{fluctuation}}} - \underbrace{\frac{1}{2}\rho V^2}_{\substack{\text{2nd Order}\\ \text{term}}}}_{\text{DynamicPressure}} - \underbrace{\rho g z}_{\substack{\text{hydrostatic}\\ \text{pressure}}} \tag{7.34}$$

Since we are only considering the case for LINEAR free surface waves we can neglect all higher order terms. Dropping the second order term in the dynamic pressure from equation (7.34), the pressure under a wave is simply

$$p_d(x,z,t) = -\rho\frac{\partial\phi}{\partial t} \tag{7.35}$$

$$= \frac{a\omega^2}{k}\rho\ f(z)\cos(\omega t - kx - \psi) \tag{7.36}$$

$$= \rho\frac{\omega^2}{k}\ f(z)\,\eta(x,t)\,, \tag{7.37}$$

where

$$\frac{\omega^2}{k}\ f(z) = g\ \tanh(kH)\frac{\cosh[k(z+H)]}{\sinh(kH)} = g\ \frac{\cosh[k(z+H)]}{\cosh(kH)}. \tag{7.38}$$

Therefore the dynamic pressure for all depths becomes

$$\boxed{p_d(x,z,t) = \rho g\,\eta(x,t)\,\frac{\cosh[k(z+H)]}{\cosh(kH)}} \tag{7.39}$$

V. Motion of Fluid Particles below a Wave

We can define the motion of fluid particles underneath a linear progressive wave to have a horizontal motion, ζ_p and a vertical motion, η_p , such that

$$\zeta_p(x,z,t) = a\ f(z)\sin(kx-\omega t)+\psi) \tag{7.40}$$

$$\eta_p(x,z,t) = a\ f_1(z)\cos(\omega t - k - x + \psi) \tag{7.41}$$

The orbital pattern is given by the equation for an ellipse:

$$\left[\frac{\zeta_p}{f(z)}\right]^2 + \left[\frac{\zeta_p}{f_1(z)}\right]^2 = a^2 \tag{7.42}$$

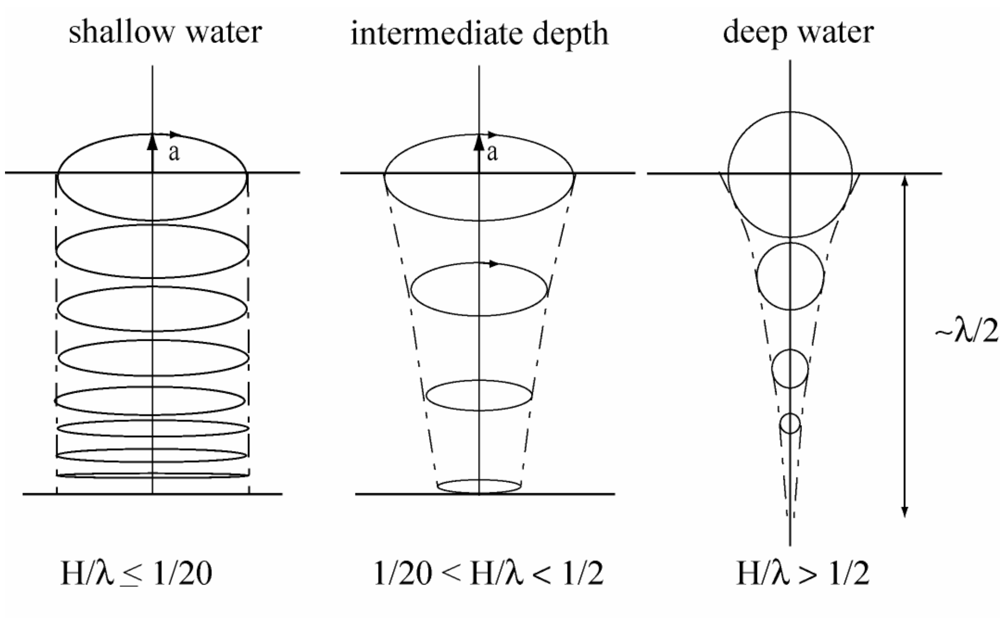
where *a* is the amplitude of the waves. Plugging in *f(z)* and *f_1(z)* from equations (7.26) and (7.27) we get

$$\left[\frac{\zeta_p}{a\left(\frac{\cosh k(z+H)}{\sinh kH}\right)}\right]^2+\left[\frac{\zeta_p}{a\left(\frac{\sinh k(z+H)}{\sinh kH}\right)}\right]^2=1 \quad (7.43)$$

The horizontal and vertical velocity components, *u* and *w*, of these particles are simply the time derivatives of the motions:

$$u=\frac{\partial \zeta_p}{\partial t} \quad (7.44)$$

$$w=\frac{\partial \eta_p}{\partial t} \quad (7.45)$$



3. Fluid particles move in elliptical orbits below the surface. In deep water, as $H\to\infty$, these ellipses become circular.

Deep Water Simplification

In deep water as $H\to\infty$, then $f(z)\approx f_1(z)\approx e^{kz}$, and the particle orbits become circular with an exponentially decreasing radius. All particle motion dies out at $z\approx-\lambda/2$.

Phase and Group Velocity

Phase Speed

The phase speed of a wave is defined as the speed at which the wave is moving. If you were to clock a wave crest you would find that it moves at the phase speed, $C_p = \omega/k$.

Group Velocity

Group velocity is the speed of propagation of a packet, or group, of waves. This is always slower than the phase speed of the waves. In a laboratory setting group speed can be observed by creating a short packet of waves, about 8-10 cycles, and observing this packet as it propagates down a testing tank. The leading edge of the packet will appear to move slower than the waves within the packet. Individual waves will appear at the rear of the packet and propagate to the front, where the pressure forces their apparent disappearance.

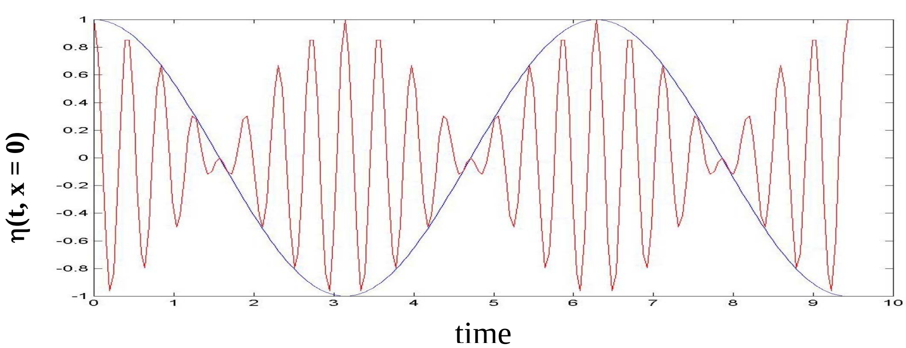
We can **derive Group Velocity** starting with a harmonic surface wave with surface elevation

$$\eta(x,t) = a\cos(\omega t - kx) \tag{7.46}$$

and then looking at that wave as if it was the sum of two cosines with very similar frequencies ($\omega \pm \delta\omega$) and wavenumbers ($k \pm \delta k$):

$$\eta(x,t) = \lim_{\delta k,\delta\omega \to 0} \left\{ \tfrac{a}{2}\cos\left([\omega - \delta\omega]t - [k - \delta k]x\right) + \tfrac{a}{2}\cos\left([\omega + \delta\omega]t - [k + \delta k]x\right) \right\} \tag{7.47}$$

$$= \lim_{\delta k,\delta\omega \to 0} \left\{ a\cos(\omega t - kx)\cos(\delta\omega\, t - \delta k\, x) \right\} \tag{7.48}$$



4. A wave train (red line) constructed with two waves having similar frequency/wavenumber components. This wave appears to be beating. The blue line shows the “envelope” or group outline.

In general the envelope (or “Group") of the wave travels at the group speed, C_g,

$$C_g = \frac{\delta\omega}{\delta k}. \tag{7.49}$$

In the limit as $\delta\omega \to 0$ and $\delta k \to 0$, the group velocity becomes the derivative of the frequency with respect to the wavenumber:

$$C_g = \frac{d\omega}{dk} \tag{7.50}$$

This can be found using the dispersion relationship from equation (7.30), and then taking the derivative of it with respect to the wavenumber.

We can derive the group speed as follows:

$$\frac{d}{dk}\left\{\omega^2\right\} = \frac{d}{dk}\left\{kg\ \tanh(kH)\right\} \tag{7.51}$$

$$2\omega\frac{d\omega}{dk} = g\ \tanh(kH) + \frac{kgH}{\cosh^2(kH)} \tag{7.52}$$

$$\frac{d\omega}{dk} = \frac{1}{2}\underbrace{\frac{g}{\omega}\tanh(kH)}_{=\omega/k=C_p}\left\{1+\frac{kH}{\sinh(kH)\cosh(kH)}\right\} \tag{7.53}$$

$$\boxed{\therefore C_g = \frac{1}{2}C_p\left\{1+\frac{kH}{\sinh kH\ \cosh kH}\right\}} \tag{7.54}$$

Deep Water: $H \to \infty$

$$\omega^2 = kg \text{ and } C_g = \tfrac{1}{2}C_p \tag{7.55}$$

Shallow Water: $H \to 0$

$$\omega = \sqrt{gH}\,k \text{ and } C_g = C_p \tag{7.56}$$

Wave Energy

- **Potential Energy**

$$E_P = \frac{1}{\lambda}\int_0^{\lambda} \frac{1}{2}\rho g \eta^2 \, dx = \tfrac{1}{\lambda}\int_0^{\lambda} \tfrac{1}{2}\rho g a^2 \cos^2(\omega t - kx) \tag{7.57}$$

$$E_P = \frac{1}{4}\rho g a^2 \tag{7.58}$$

- **Kinetic Energy**

$$E_k = \frac{1}{\lambda}\int_0^{\lambda}\int_{-H}^{0} \frac{1}{2}\rho(u^2 + w^2)\, dz\, dx \tag{7.59}$$

$$E_k = \frac{1}{4}\rho g a^2 \tag{7.60}$$

- **Total Energy per wavelength**

$$\overline{E} = E_P + E_k = \frac{1}{2}\rho g a^2 \tag{7.61}$$

Flux of Energy through a Vertical Plane

$$Power = force * velocity = (p\ dz)u$$

- Energy flux:

$$\frac{dE}{dt} = \int_{-H}^{0} p\, u\, dz \tag{7.62}$$

$$= \frac{1}{2k}\rho g a^2 \omega \cos^2(\omega t - kx) \tag{7.63}$$

- Average energy flux over one cycle, assuming deep water,

$$\frac{d\overline{E}}{dt} = \frac{1}{T}\int_0^T \frac{dE}{dt} dt = \frac{1}{4k}\rho g a^2 \omega = \overline{E} \cdot C_g \tag{7.64}$$

Useful References

The material covered in this section should be review. If you have not taken 13.021 or a similar class dealing with basic fluid mechanics and water waves please contact the instructor. The references below are merely suggestions for further reading and reference.

- J. N. Newman (1977) *Marine Hydrodynamics* MIT Press, Cambridge, MA.
- M. Faltinsen (1990) *Sea Loads on Ships and Offshore Structures* Cambridge University Press, Cambridge, UK.
- M. Rahman (1995) *Water Waves: Relating modern theory to advanced engineering practice* Clarendon Press, Oxford.

