2.003J/1.053J Dynamics and Control I, Spring 2007
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2/21/2007

Lecture 5

Systems of Particles: Example 1: Linear Momentum and Conservation of Energy, Example 2: Angular Momentum

Systems of Particles Examples

Example 1: Block Sliding Down an Inclined Plane (continued)

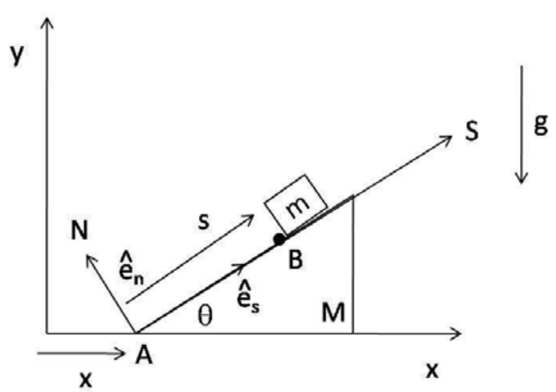

Figure 1: Kinematic diagram of block sliding on ramp. The block rests on a frictionless surface and the ramp also acts on a frictionless surface as well. Figure by MIT OCW.

Initially released with $s = l$.

$\underline{r}_A = x\hat{\imath} \quad \underline{r}_B = x\hat{\imath} + s\hat{e}_s = x\hat{\imath} + s\cos\theta\hat{\imath} + s\sin\theta\hat{\jmath}$

$\dot{\underline{r}}_A = \dot{x}\hat{\imath} \quad \dot{\underline{r}}_B = \dot{x}\hat{\imath} + \dot{s}\cos\theta\hat{\imath} + \dot{s}\sin\theta\hat{\jmath}$

$\ddot{r}_A = \ddot{x}\hat{\imath} \quad \underline{\ddot{r}_B} = \ddot{x}\hat{\imath} + \ddot{s}\cos\theta\hat{\imath} + \ddot{s}\sin\theta\hat{\jmath}$

What is the velocity of M at the moment m reaches point A?

Free Body Diagrams (FBDs)

System:

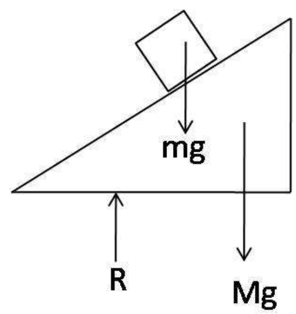

Figure 2: Free body diagram of block on ramp. The forces present are a reaction force, R, and the forces due to gravity on both bodies, Mg and mg. Figure by MIT OCW.

Separately:

Because we are not trying to calculate each force, apply linear momentum principle so that N does not appear. Use system.

Linear momentum principle

To system:

$$\underline{F} = \frac{d}{dt}\underline{P}$$

No forces in x-direction. $(\underline{F})_x = 0 \Rightarrow$ Linear momentum in the x-direction is conserved.

$$M\dot{x} + m(\dot{x} + \dot{s}\cos\theta) = \text{constant} = 0 \tag{1}$$

constant = 0 because system initially at rest.

We recognize that our system momentum equation is a consequence of taking each individual equation together.

We have two variables x and s. So we need 2 equations using $\dot{x}$ and $\dot{s}$. (An expression for velocities).
Note: For individual masses,

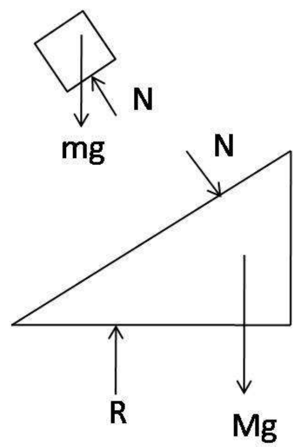

Figure 3: Free body diagram of block on ramp analyzed seperately as individual components. A normal force N acts on both bodies. The force due to gravity mg acts on the block and the force due to gravity Mg acts on the ramp. Figure by MIT OCW.

$$\frac{d}{dt}(M\dot{x}) = N\sin\theta$$

$$\frac{d}{dt}(m(\dot{x} + \dot{s}\cos\theta)) = -N\sin\theta$$

Already used linear momentum, now must either choose conservation of energy or angular momentum. Try conservation of energy first.

We can use conservation of energy because all external forces are potential or do no work. (No friction, gravity, R does no work because $\perp$ to motion).
And interntal forces (N) do no work.

Conservation of Energy

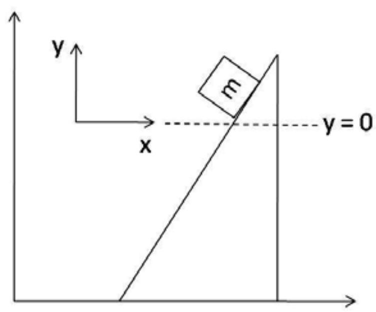

Figure 4: Diagram depicting conservation of energy. Block is initially at rest at $y = 0$. Figure by MIT OCW.

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 0 = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[(\dot{x} + \dot{s}\cos\theta)^2 + \dot{s}^2\sin^2\theta] - mg(l - s)\sin\theta$$

$\frac{1}{2}m[(\dot{x} + \dot{s}\cos\theta)^2 + \dot{s}^2\sin^2\theta]$: Kinetic Energy
$-mg(l - s)\sin\theta$: Potential Energy

This is the general relation for the system in any configuration after the mass m has been released from rest at $y = 0$.

Combine Linear Momentum Principle and Conservation of Energy

Our particular interest is for $s = 0$.

$$\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[(\dot{x} + \dot{s}\cos\theta)^2 + \dot{s}^2\sin^2\theta] - mgl\sin\theta = 0$$

$$\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[\dot{x}^2 + 2\dot{x}\dot{s}\cos\theta + \dot{s}^2\cos^2\theta + \dot{s}^2\sin^2\theta] - mgl\sin\theta = 0$$

$$\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{x}\dot{s}\cos\theta = mgl\sin\theta \tag{2}$$

From Equation 1,

$$\dot{s} = \frac{-(M + m)}{m\cos\theta}\dot{x}$$

Substitute in Equation 2:

$$\frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}\frac{(M+m)}{m\cos^2\theta}\dot{x}^2 + \dot{x}^2(-(M+m)) = mgl\sin\theta$$

Rewritten:

$$\frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}\frac{(M+m)^2\dot{x}^2}{m\cos^2\theta} - (M+m)\dot{x}^2 = mgl\sin\theta$$

$$\left[\frac{1}{2}\frac{(M+m)^2}{m\cos^2\theta} - \left(\frac{1}{2}(M+m)\right)\right]\dot{x}^2 = mgl\sin\theta$$

$$\dot{x} = \left[\frac{mgl\sin\theta}{\frac{1}{2}\frac{(M+m)^2}{m\cos^2\theta} - \frac{1}{2}(M+m)}\right]^{\frac{1}{2}}$$

1. Is there an optimum angle for the greatest velocity?
Take derivative with respect to θ and set to 0.

$$\frac{d\dot{x}}{d\theta} = 0$$

Check minimum or maximum.

2. If one divides the top and bottom by m, one obtains an expression for $\dot{x}$ in terms of θ and the mass ratio $\frac{M}{m}$. To optimize with respect to mass ratio:

$$\frac{d\dot{x}}{d\left(\frac{M}{m}\right)} = 0$$

Check minimum or maximum.

3. Knowing $\dot{x}$, we know $\dot{s}$. To get forces N, needs more work. Differentiate Equation 2 to find acceleration.

Example 2: Cart With Spring and Pendulum

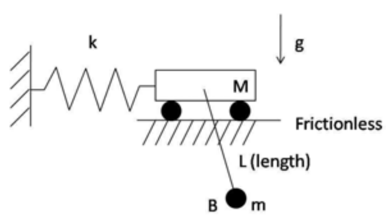

Figure 5: Cart with spring and pendulum. Figure by MIT OCW.

Derive the equations of motion for this system. A is the center of mass for this uniform block. The spring is attached along a line passing through A. 2 degrees of freedom. 2 coordinate systems.

Kinematics

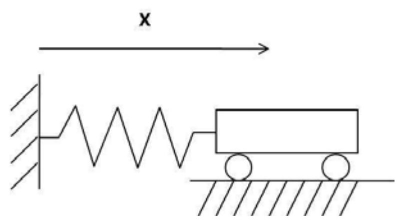

Figure 6: Kinematic diagram of cart with spring and pendulum. Figure by MIT OCW.

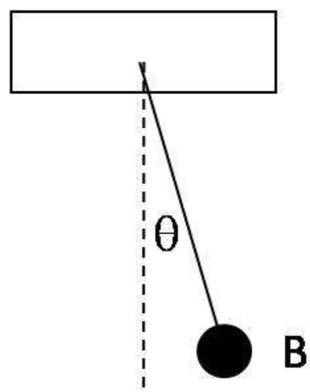

Figure 7: Kinematic diagram of pendulum. Figure by MIT OCW.

Use x for horizontal position. String is unstretched when $x = 0$.

$$r_A = x\hat{\imath} \quad r_B = (x + L\sin\theta)\hat{\imath} - L\cos\theta\hat{\jmath}$$
$$\dot{r}_A = \dot{x}\hat{\imath} \quad \dot{r}_B = (\dot{x} + L\dot{\theta}\cos\theta)\hat{\imath} + L\dot{\theta}\sin\theta\hat{\jmath}$$
$$\ddot{r}_A = \ddot{x}\hat{\imath} \quad \ddot{r}_B = (\ddot{x} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta)\hat{\imath} + (L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta)\hat{\jmath}$$

Free Body Diagrams (FBDs)

Use system approach. Internal forces of string on block do not turn up.

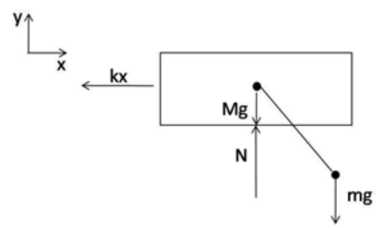

Figure 8: Free body diagram of cart with spring and pendulum. Figure by MIT OCW.

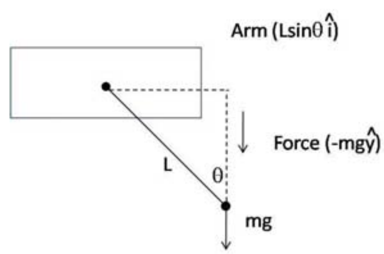

Figure 9: Free body diagram of pendulum arm. Figure by MIT OCW.

Forces (Kinetics)

Linear Momentum Principle

$$\begin{aligned} \underline{F} &= \frac{d}{dt}\underline{P} \\ &= \frac{d}{dt}(M\dot{x} + m(\dot{x} + L\dot{\theta}\cos\theta)) \end{aligned}$$

$$\boxed{-kx = M\ddot{x} + m\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta}$$

We need another equation because we have two variables. Could consider conservation of energy, but instead, we will use the principle of angular momentum.

Angular Momentum Principle Apply about point A. Any forces at A will generate no torque. N generated no torque either because it acts through center of mass A.

Angular Momentum about A

$$\underline{\tau}_A = \frac{d}{dt}\underline{H}_A + \underline{v}_A \times \underline{P}$$

$$\underline{\tau}_A = -mgL\sin\theta\hat{k}$$

$$\begin{aligned}
\underline{H}_A &= \underline{AB} \times m\dot{r}_B \\
&= (L\sin\theta\hat{\imath} - L\cos\theta\hat{\jmath}) \times m[(\dot{x} + L\dot{\theta}\cos\theta)\hat{\imath} + (L\dot{\theta}\sin\theta)]\hat{\jmath} \\
&= (mL^2\dot{\theta} + mL\dot{x}\cos\theta)\hat{k}
\end{aligned}$$

$$\begin{aligned}
\underline{v}_A \times \underline{P} &= \dot{x}\hat{\imath} \times (M\dot{x}\hat{\imath} + m(\dot{x} + L\dot{\theta}\cos\theta)\hat{\imath} + m(L\dot{\theta}\sin\theta)\hat{\jmath}) \\
&= mL\dot{x}\dot{\theta}\sin\theta\hat{k}
\end{aligned}$$

See Recitation 2 for the rest of Example 2 and an outline of numerical solution with MATLAB.