Our goal for this lecture is to prove that morphisms of projective varieties are closed maps. In fact we will prove something stronger, that projective varieties are *complete*, a property that plays a role comparable to compactness in topology. For varieties, compactness as a topological space does not mean much because the Zariski topology is so coarse. Indeed, *every* subset of $\mathbb{A}^n$ (and hence of $\mathbb{P}^n$) is compact (or quasicompact, if your definition of compactness includes Hausdorff).

Theorem 16.1. *Let S be a subset of $\mathbb{A}^n$, and let $\{U_a\}_{a\in A}$ be any collection of open sets of $\mathbb{A}^n$ whose union contains S. Then there exists a finite set $B \subseteq A$ for which $S \subseteq \{U_b\}_{b\in B}$.*

Proof. By enumerating the index set A in some order (which we can do, via the axiom of choice), we can construct a chain of properly nested open sets $\{V_b\}_{b\in B}$, where each V_b is the union of the sets U_a over $a \in B$ with $a \le b$ (in our arbitrary ordering), and $B \subseteq A$ is constructed so that each $S \cap V_a$ is properly contained in $S \cap V_b$ for every pair $a \le b$ in B. The complements of the sets V_b then form a strictly descending chain of closed sets whose ideals form a strictly ascending chain of nested ideals $\{I_b\}_{b\in B}$ in $R = k[x_1,\ldots,x_n]$. The ring R is Noetherian, so B must be finite, and $\{U_b\}_{b\in B}$ is the desired finite subcover. $\square$

In order to say what it means for a variety to be complete, we first need to define the product of two varieties. Throughout this lecture k denotes a fixed algebraically closed field.

16.1 Products of varieties

Definition 16.2. Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be algebraic sets. Let $k[\mathbb{A}^m] = k[x_1,\ldots,x_m]$, $k[\mathbb{A}^n] = k[y_1,\ldots,y_n]$, and $k[\mathbb{A}^{m+n}] = k[x_1,\ldots,x_m,y_1,\ldots,y_n]$, so that we can identity $k[\mathbb{A}^m]$ and $k[\mathbb{A}^n]$ as subrings of $k[\mathbb{A}^{m+n}]$ whose intersection is k. The *product* $X \times Y$ is the zero locus of the ideal $I(X)k[\mathbb{A}^n] + I(Y)k[\mathbb{A}^m]$ in $k[\mathbb{A}^{m+n}]$.

If $I(X) = (f_1,\ldots,f_s)$ and $I(Y) = (g_1,\ldots,g_t)$, then $I(X\times Y) = (f_1,\ldots,f_s,g_1,\ldots,g_t)$ is just the ideal generated by the f_i and g_j when regarded as elements of $k[\mathbb{A}^{m+n}]$. We also have *projection morphisms*

$$\pi_X\colon X\times Y \to X \quad\text{ and }\quad \pi_Y\colon X\times Y\to Y$$

defined by the tuples $(\bar{x}_1,\ldots,\bar{x}_m)$ and $(\bar{y}_1,\ldots,\bar{y}_n)$, where $\bar{x}_i$ and $\bar{y}_j$ are the images of x_i and y_j, respectively, under the quotient map $k[\mathbb{A}^{m+n}] \to k[\mathbb{A}^{m+n}]/I(X\times Y) = k[X\times Y]$.

The coordinate ring of $X\times Y$ is isomorphic to the *tensor product* of the coordinate rings of X and Y, that is

$$k[X\times Y] \simeq k[X]\otimes k[Y].$$

While the tensor product can be defined quite generally in categorical terms, in the case of k-algebras there is a very simple concrete definition. Recall that a k-algebra is, in particular, a k-vector space. If R and S are two k-algebras with bases $\{r_i\}_{i\in I}$ and $\{s_j\}_{j\in J}$, then the set of formal symbols $\{r_i\otimes s_j : i\in I, j\in J\}$ forms a basis for the tensor product $R\otimes S$. Products of vectors in $R\otimes S$ are computed via the distributive law and the rule

$$(r_{i_1}\otimes s_{j_1})(r_{i_2}\otimes s_{j_2}) = r_{i_1}r_{i_2}\otimes s_{j_1}s_{j_2}.$$

Andrew V. Sutherland

In the case of polynomial rings one naturally chooses a monomial basis, in which case this rule just amounts to multiplying monomials and keeping the variables in the monomials separated according to which polynomial ring they originally came from.

It is standard to generalize the $\otimes$ notation and write $r \otimes s$ for any $r \in R$ and $s \in S$, not just basis elements, with the understanding that $r \otimes s$ represents a linear combination of basis elements $\sum_{i,j} \alpha_{ij}(r_i \otimes s_j)$ that can be computed by applying the identities

$$\begin{aligned}(a+b)\otimes c &= a\otimes c + b\otimes c\\ a\otimes(b+c) &= a\otimes b + a\otimes c\\ (\gamma a)\otimes(\delta b) &= (\gamma\delta)(a\otimes b)\end{aligned}$$

where γ and δ denote elements of the field k. We should note that most elements of $R \otimes S$ are *not* of the form $r \otimes s$, but they can all be written as finite sums of elements of this form.

When R and S are commutative rings, so is $R \otimes S$. There are then natural embeddings of R and S into $R \otimes S$ given by the maps $r \to r \otimes 1_S$ and $s \to 1_R \otimes s$, and $1_R \otimes 1_S$ is the multiplicative identity in $R \otimes S$. The one additional fact that we need is that if R and S are affine algebras (finitely generated k-algebras that are integral domains), so is $R \otimes S$. In order to prove this we first note a basic fact that we will use repeatedly:

Lemma 16.3. *Let V be an affine variety with coordinate ring $k[V]$. There is a one-to-one correspondence between the maximal ideals of $k[V]$ and the points of V.*

Proof. Let $P = (a_1, \ldots, a_n)$ be a point on $V \subseteq \mathbb{A}^n$, and let m_P be the corresponding maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$ of $k[x_1, \ldots, x_n]$. Then $I(V) \subseteq m_P$, and the image of m_P in the quotient $k[V] = k[x_1, \ldots, x_n]/I(V)$ is a maximal ideal of $k[V]$. Conversely, every maximal ideal of $k[V]$ corresponds to a maximal ideal of $k[x_1, \ldots, x_n]$ that contains $I(V)$, which is necessarily of the form m_P for some $P \in V$, by Hilbert's Nullstellensatz. □

Lemma 16.4. *If R and S are both affine algebras, then so is $R \otimes S$.*

Proof. We need to show that $R \otimes S$ has no zero divisors. So suppose $uv = 0$ for some $u, v \in R \otimes S$. We will show that either $u = 0$ or $v = 0$.

We can write u and v as finite sums $u = \sum_{i \in I} r_i \otimes s_i$ and $v = \sum_{j \in J} r_j \otimes s_j$, with $r_i, r_j \in R$ and $s_i, s_j \in S$ all nonzero, and we can assume the sets $\{s_i\}_{i \in I}$ and $\{s_j\}_{j \in J}$ are each linearly independent over k by choosing the s_i and s_j to be basis vectors. Without loss of generality, we may assume $R = k[X]$, for some affine variety X. Let X_u be the zero locus of the r_i in X and and let X_v be the zero locus of the r_j in X. For any point $P \in X$ we have the evaluation map $\phi_P \colon k[X] \to k$ defined by $\phi_P(f) = f(P)$, which is a ring homomorphism from R to k that fixes k. We now extend ϕ_P to a k-algebra homomorphism $R \otimes S \to S$ by defining $\phi_P(r \otimes s) = \phi_P(r)s$. We then have

$$\phi_P(uv) = \phi_P(u)\phi_P(v) = \left(\sum_{i \in I} \phi_P(r_i)s_i\right)\left(\sum_{j \in J} \phi_P(r_j)s_j\right) = 0$$

Since S is an integral domain, one of the two sums must be zero, and since the s_i are linearly independent over k, either $\phi_P(r_i) = 0$ for all the r_i, in which case $P \in X_u$, or $\phi_P(r_j) = 0$ for all the r_j, in which case $P \in X_v$. Thus $X = X_u \cup X_v$. But X is irreducible, so either $X = X_u$, in which cace $u = 0$, or $X = X_v$, in which case $v = 0$. □

Corollary 16.5. *If X and Y are affine varieties, then so is $X \times Y$.*

Remark 16.6. This proof is a nice example of the interaction between algebra and geometry. We want to prove a geometric fact (a product of varieties is a variety), but it is easier to prove an algebraic fact (a tensor product of affine algebras is an affine algebra). But in order to prove the algebraic fact, we use a geometric fact (a variety is not the union of two proper algebraic subsets). Of course we could translate everything into purely algebraic or purely geometric terms, but the proofs are easier to construct (and easier to understand!) when we can move back and forth freely.

A product of projective varieties is defined similarly, but there is a new wrinkle; we now need two distinct sets of homogeneous coordinates. Points in $\mathbb{P}^m \times \mathbb{P}^n$ can be represented in the form $(a_0 : \ldots : a_m; b_0 : \ldots : b_n)$, where

$$(a_0 : \ldots : a_m; b_0 : \ldots : b_n) = (\lambda a_0 : \ldots : \lambda a_m; \mu b_0 : \ldots : \mu b_n)$$

for all $\lambda, \mu \in k^\times$. We are now interested in polynomials in $k[x_0, \ldots, x_m, y_0, \ldots, y_n]$ that are homogeneous in the x_i, and in the y_j, but not necessarily both. Another way of saying this is that we are interested in polynomials that are homogeneous as elements of $(k[x_0, \ldots, x_m])[y_0, \ldots, y_n]$, and as elements of $(k[y_0, \ldots, y_n])[x_0, \ldots, x_m]$. Let us call such polynomials (m, n)-*homogeneous*. We can then meaningfully define the zero locus of an (m, n)-homogeneous polynomial in $\mathbb{P}^m \times \mathbb{P}^n$ and give $\mathbb{P}^m \times \mathbb{P}^n$ the Zariski topology by taking algebraic sets to be closed.

Remark 16.7. The Zariski topology on $\mathbb{P}^m \times \mathbb{P}^n$ we have just defined is *not* the product of the Zariski topologies on $\mathbb{P}^m$ and $\mathbb{P}^n$. This will be explored on the problem set.

Definition 16.8. Let $X \subseteq \mathbb{P}^m$ and $Y \subseteq \mathbb{P}^n$ be algebraic sets with homogeneous ideals $I(X) \subseteq k[x_0, \ldots, x_m]$ and $I(Y) \subseteq k[y_0, \ldots, y_n]$. The *product* $X \times Y$ is the zero locus of the (m, n)-homogeneous polynomials in the ideal

$$I(X \times Y) := I(X)k[y_0, \ldots, y_n] + I(Y)k[x_0, \ldots, x_m]$$

of $k[x_0, \ldots, x_m, y_0, \ldots, y_n]$. We say that $X \times Y$ is a *variety* if the ideal $I(X \times Y)$ is prime.

As in the affine case, we again have $k[X \times Y] = k[X] \otimes k[Y]$, which implies that the product of two projective varieties is again a variety.

Remark 16.9. One can identify $\mathbb{P}^m \times \mathbb{P}^n$ with a subvariety of a larger projective space $\mathbb{P}^N$ (but N is definitely not $m + n$). Thus the product of two projective varieties is indeed a projective variety. This will be explored on the next problem set.

We may also consider products of affine and projective varieties. In this case we are interested in subsets of $\mathbb{P}^m \otimes \mathbb{A}^n$ that are the zero locus of polynomials in $k[x_0, \ldots, x_m, y_1, \ldots, y_n]$ that are homogeneous in x_i but may be inhomogeneous in the y_j. Per the remark above, we can smoothly embed a product of projective varieties in a single projective variety, and as we have already seen we can smoothly embed a product of affine varieties in a single affine variety. Thus any finite product of affine and projective varieties is isomorphic to one of (1) an affine variety, (2) a projective variety, (3) the product of a affine variety and a projective variety.

16.2 Complete varieties

We can now say what it means for a variety to be complete.

Definition 16.10. A variety X is *complete* if for every variety Y the projection $X \times Y \to Y$ is a *closed map*; this means that the projection of a closed set in $X \times Y$ is a closed in Y.

Remark 16.11. We get the same definition if we restrict to affine varieties Y. Any variety Y can be covered by a finite number of affine parts $\{U_i\}$, and if the projection $X \times U_i \to U_i$ is a closed map for each U_i, then the projection $X \times Y \to Y$ is also a closed map, since the union of a finite number of closed sets is a closed set.

Lemma 16.12. *If X is a complete variety then any morphism $\phi\colon X \to Y$ is a closed map whose image is a complete variety.*

Proof. Let us consider the set

$$\Gamma_\phi := \{(P, \phi(P)) : P \in X\} \subseteq X \times Y,$$

which is the *graph* of ϕ. It is a closed set, the zero locus of $y = \phi(x)$ (here the variables x and y represent points in X and Y that may have many coordinates; the exact equation can be explicitly spelled out in the ambient space containing $X \times Y$ using generators for $I(X)$, $I(Y)$, and the coordinate maps of ϕ but there is no need to do so). The projection map $X \times Y \to Y$ is a closed map, since X is complete, so $\operatorname{im}(\phi)$ is a closed subset of Y, and it must be irreducible, since it is the image of a variety. Similarly, if Z is any closed set in X, by considering the graph of the restriction of ϕ to Z and applying the fact that X is complete we can show that $\phi(Z)$ is closed. Thus ϕ is a closed map.

We now show that $\phi(X)$ is complete. So let Z be any variety and consider the projection $\phi(X) \times Z \to Z$. Let us define the morphism $\Phi\colon X \times Z \to Y \times Z$ by $\Phi(P, Q) = (\phi(P), Q)$. If V is a closed set in $\phi(X) \times Z \subseteq Y \times Z$, then its inverse image $\Phi^{-1}(V)$ is closed in $X \times Z$, since Φ is continuous. Since X is complete, the projection of $\Phi^{-1}(V)$ to Z is closed, but this is precisely the projection of V to Z, since the Z-component of Φ is the identity map. $\square$

Lemma 16.13. *If X is complete then so is every subvariety of X.*

Proof. Let $V \subseteq X$ be a variety. For any variety Z the projection $V \times Z \to Z$ is the composition

$$V \times Z \to X \times Z \to Z,$$

where the first map is an inclusion and the second map is a projection, both of which are closed maps. Thus the projection $V \times Z \to Z$ is a closed map and V is complete. $\square$

Theorem 16.14. *Every complete affine variety consists of a single point.*

Proof. We first consider $\mathbb{A}^1$ and the closed set $\{(x, y) : xy = 1\}$ in $\mathbb{A}^1 \times \mathbb{A}^1$. The projection to the second $\mathbb{A}^1$ is $\mathbb{A}^1 - \{0\}$, not a closed set, so the first $\mathbb{A}^1$ is not complete.

Now suppose X is an affine variety of positive dimension and let f be a function in $k[X]$ that does not lie in k; such an f exists since $k(X)$ has positive transcendence degree. The morphism $f\colon X \to \mathbb{A}^1$ that sends P to $f(P)$ most then be dominant, because the dual morphism of affine algebras $k[\mathbb{A}^1] \to k[X]$ is injective; it corresponds to the inclusion $k[f] \subseteq k[X]$ with $k \subsetneq k[f]$. But X is complete, so by Lemma 16.12 the image of $f\colon X \to \mathbb{A}^1$ is a complete variety, and f is dominant, so $\mathbb{A}^1$ is complete, a contradiction.

Thus every complete affine variety has dimension 0 and is therefore a point. $\square$

With one trivial exception, affine varieties are not complete. In contrast, we will prove that every projective variety is complete.

In order to prove this we will apply a theorem of Chevalley that gives a criterion for the completeness of a variety in terms of the *valuation rings* in the function fields of all its subvarieties; this is known as the *valuative criterion* for completeness. But we first take a brief interlude to discuss valuation rings.

16.3 Valuation rings

We have already seen many examples of valuation rings in this course, but let us now formally define the general term.

Definition 16.15. A proper subring R of a field K is a *valuation ring* of K if for every $x \in K^\times$, either $x \in R$ or $x^{-1} \in R$ (possibly both).

Note that a valuation ring R is an integral domain (since it is a subring of a field), and that K is its field of fractions. Given an arbitrary integral domain R that is not a field, we say that R is a valuation ring if it is a valuation ring of its fraction field. In Problem Set 2 you proved that if K is any field with an nonarchimedean absolute value $\| \ \|$, then the set

$$R = \{x \in K : \|x\| \le 1\}$$

is a valuation ring. You also proved that such an R is a *local ring*.

Definition 16.16. A *local ring* is a ring R with a unique maximal ideal $\mathfrak{m}$. The field $R/\mathfrak{m}$ is the *residue field* of R.

Note that fields are included in the definition of a local ring (the unique maximal ideal is the zero ideal), but specifically excluded from the definition of a valuation ring.

Lemma 16.17. *A ring R is a local ring if and only if the set $R - R^\times$ is an ideal.*

Proof. If $R - R^\times$ is an ideal, then it contains every proper ideal and is therefore the unique maximal ideal of R. Conversely, every element of $R - R^\times$ lies in a maximal ideal, and if there is only one such ideal it must equal $R - R^\times$. □

Theorem 16.18. *Every valuation ring is a local ring.*

Proof. Let R be a valuation ring and let $\mathfrak{m} = R - R^\times$. We must show that $\mathfrak{m}$ is an ideal. If $a \notin R^\times$ then $ar \notin R^\times$ for all $r \in R$. So $\mathfrak{m}R \subseteq \mathfrak{m}$. If $a, b \in \mathfrak{m}$ then a/b or b/a lies in R. So $(a/b + 1)b = a + b$ or $(b/a + 1)a = b + a$ lies in $\mathfrak{m}$, hence $\mathfrak{m}$ is an ideal. □

A key property of valuation rings is that their ideals are totally ordered.

Lemma 16.19. *If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of a valuation ring R then either $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathfrak{b} \subseteq \mathfrak{a}$.*

Proof. Suppose not. Then there exist $a \in \mathfrak{a} - \mathfrak{b}$ and $b \in \mathfrak{b} - \mathfrak{a}$, both nonzero. Either a/b or b/a lies in R, so either $(a/b)b = a \in \mathfrak{b}$ or $(b/a)a = b \in \mathfrak{a}$, both of which are contradictions. □

The proof of Lemma 16.19 allows us to compare nonzero elements of R: we have $a/b \in R$ if and only if $(a) \subseteq (b)$. This leads to the following definition.

Definition 16.20. Let R be a valuation ring with fraction field K. The *value group* of R is $\Gamma = K^\times/R^\times$. The *valuation* defined by R is the quotient map $v\colon K^\times \to \Gamma$.

The abelian group Γ is typically written additively, and it follows from Lemma 16.19 that it is totally ordered (its elements are associate classes and their inverses). We have

1. $v(x) = 0$ if and only if $x \in R^\times$,
2. $v(xy) = v(x) + v(y)$,
3. $v(x + y) \geq \min(v(x), v(y))$.

The first two properties are immediate from the definition; the third will be proved on the problem set. For $x \in K^\times$ we then have $v(x) \geq 0$ if and only if x is a nonzero element of R. By convention we extend v to K by defining $v(0) = \infty$, where ∞ is greater than every element of the valuation group Γ. We then have $R = \{x \in K : v(x) \geq 0\}$.[1]

When a valuation ring R is a PID, it is then a UFD with a unique (up to associates) prime element p that generates its maximal ideal. In this case $\Gamma \simeq \mathbb{Z}$, since for nonzero $a \in R$ we can associate $v(a)$ to the largest integer n for which $p^n|a$; this also determines $v(1/a) = -v(a)$. In this situation we say that Γ is *discrete* and call R a *discrete valuation ring*. Recall that earlier we defined discrete valuation rings as local rings that are PIDs but not fields. We will see show that this definition is equivalent, and also precisely characterize the distinctions in the inclusions

$$\text{discrete valuation rings} \quad \subset \quad \text{valuation rings} \quad \subset \quad \text{local rings}$$

Lemma 16.21. *Every finitely generated ideal of a valuation ring is principal.*

Proof. Let $(a_1, \ldots, a_n)$ be a finitely generated ideal of a valuation ring R with n minimal and suppose $n > 1$. We must have $a_1/a_2 \notin R$, else the generator $a_1 = (a_1/a_2)a_2$ is redundant. But then $a_2/a_1 \in R$ and $a_2 = (a_2/a_1)a_2$ is redundant, a contradiction. $\square$

Lemma 16.22. *A local ring is a valuation ring if and only if it is an integral domain that is not a field and all of its finitely generated ideals are principal.*

Proof. The "only if" part of the statement is clear, so let us assume that R is a local ring that satisfies the hypothesis on the right, and let a/b be any element of its fraction field. The ideal (a, b) is finitely generated, hence principal, say $(a, b) = (c)$. Thus for some $d, e, f, g \in R$ we have $a = cd$, $b = ce$, and $c = af + bg = cdf + ceg$, and therefore $df + eg = 1$. If neither d nor e is a unit, then they both lie in the maximal ideal of R and so does 1, a contradiction. So one of d or e is a unit, and therefore one of $a/b = d/e$ and $b/a = e/d$ lies in R. $\square$

The second lemma implies, in particular, that our two definitions of discrete valuation ring are equivalent. Together the two lemmas give a third definition.

Corollary 16.23. *A valuation ring is discrete if and only if it is Noetherian.*

When the fraction field K of a valuation ring R is an extension of a smaller field k that is contained in R, we say that R is a valuation ring of the extension K/k.

[1] Note that for $\Gamma \subseteq R$ we define $\|x\| = c^{-v(x)}$ for some $c > 0$, so this agrees with $R = \{x \in K : \|x\| \leq 1\}$.

16.4 Localization of a ring at a prime

One of the main ways in which local rings arise is by *localizing* an integral domain at one of its prime ideals.

Definition 16.24. Let R be an integral domain and let $\mathfrak{p}$ be a prime ideal in R. The subring of R's fraction field defined by

$$R_{\mathfrak{p}} := \{a/b : a, b \in R, b \notin \mathfrak{p}\}$$

is called the *localization* of R at $\mathfrak{p}$.[2]

Remark 16.25. As we saw in Lecture 15, caution is needed when interpreting expressions like a/b in fraction fields of rings that are not necessarily UFDs; $R_{\mathfrak{p}}$ is a set of equivalence classes, and a/b is just one representative of a particular class. It may happen that the equivalence class a/b lies in $R_{\mathfrak{p}}$ even though $b \in \mathfrak{p}$; this occurs if $a/b = c/d$ for some $d \notin \mathfrak{p}$. We have $ad = bc$, so if $b \in \mathfrak{p}$ then either a or d lies in $\mathfrak{p}$, but it could be a and not d.

We view R as a subring of the localization $R_{\mathfrak{p}}$ via the canonical embedding $r \to r/1$.

Lemma 16.26. *The ring $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$*

Proof. This is obvious when $R_{\mathfrak{p}}$ is a UFD, but we can't assume this; however we can assume that we always pick representatives $a/b \in R_{\mathfrak{p}}$ so that $b \notin \mathfrak{p}$. If $a/b \in R_{\mathfrak{p}}$ is not in $\mathfrak{p}R_{\mathfrak{p}}$ then clearly $a \notin \mathfrak{p}$ and therefore $b/a \in R_{\mathfrak{p}}$, so a/b is a unit. Conversely, if $a/b \in R_{\mathfrak{p}}$ is a unit then $(a/b)(c/d) = 1$ for some $c, d \in R$ with $d \notin \mathfrak{p}$. We then have $ac = bd$, and if a is in $\mathfrak{p}$, then so is bd, but then either $b \in \mathfrak{p}$ or $d \in \mathfrak{p}$, since $\mathfrak{p}$ is prime, which is a contradiction. Thus $R_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}} \sqcup R_{\mathfrak{p}}^{\times}$, therefore $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. □

In general, the localization $R_{\mathfrak{p}}$ need not be a valuation ring, but provided that $\mathfrak{p}$ is nonzero it is always contained in one, as you will prove on the problem set.

16.5 Valuative criterion for completeness

We now return to our goal of proving that every projective variety is complete. Let X be a variety with coordinate ring $k[X]$, and let P be a point in X. We then define the ideal

$$m_P := \{f \in k[X] : f(P) = 0\}.$$

Note that we have defined what $f(P) = 0$ means, and even how to evaluate f at P, for all the varieties we have considered, so this definition applies to any variety, not just affine varieties. Indeed, m_P is the kernel of the evaluation map $k[X] \to k$ defined by $f \to f(P)$. This makes it clear that m_P is a maximal ideal, since the quotient $k[X]/m_P \simeq k$ is a field.

Definition 16.27. Let X be a variety with coordinate ring $k[X]$ and let $P \in X$. The *local ring of P on X* is the ring

$$\mathcal{O}_P := \mathcal{O}_{P,X} := k[X]_{m_P} = \{g/h \in k(X) : h \notin m_P\}.$$

With Remark 16.25 in mind, it is clear that $\mathcal{O}_P$ is precisely the ring of functions in $k(X)$ that are regular at P.

[2] Be sure not to confuse $R_{\mathfrak{p}}$ with the quotient $R/\mathfrak{p}$.

We are now ready to state Chevalley's valuative criterion for completeness.

Theorem 16.28. *Let X be a variety such that for every subvariety $Z \subseteq X$ and valuation ring R of $k(Z)/k$ there exists a point $P \in Z$ such that $\mathcal{O}_{P,Z} \subseteq R$. Then X is complete.*

The proof below is adapted from [2, Prop. 7.17].

Proof. So let Y be an affine variety and let $V \subseteq X \times Y$ be a closed set. We may assume that V is irreducible, since we can always write V as a finite union of irreducible sets (the coordinate ring of $X \times Y$ is Noetherian) and then prove that the image of each is closed, and we may replace Y with the image of $V \subseteq X \times Y \to Y$, since whether the image is closed or not does not depend on anything outside of its closure. We now replace X with the image Z of $V \subseteq X \times Y \to X$, to which we will apply the hypothesis of the theorem.

We have the following commutative diagram with dominant morphisms ϕ and ψ.

$$\begin{tikzcd}
 & & Y \\
V \arrow[r, "\subseteq"] \arrow[urr, bend left, "\varphi"] \arrow[drr, bend right, "\psi"'] & Z \times Y \arrow[ur, "\pi_Y"] \arrow[dr, "\pi_Z"] & \\
 & & Z
\end{tikzcd}$$

We need to show that the morphism φ is actually a surjection. So let Q be any point in Y; we will construct a point P such that (P, Q) is in V, which will prove $Q \in \varphi(V)$.

Let $\phi\colon k[Y] \to k$ be the evaluation map $\phi(g) = g(Q)$, which we note fixes k (and is therefore surjective). The morphism of affine algebras $\varphi^*\colon k[Y] \to k[V]$ is injective, since φ is dominant, thus we may regard $k[Y]$ as a subring of $k[V]$, which is in turn embedded in the function field $k(V)$. By Lemma 16.29 below, there exists a valuation ring S of $k(V)/k$ that contains the image of $k[Y]$ in $k(V)$ such that the quotient map $\Phi\colon S \to k$ from S to its residue field k is an extension of ϕ.

Let us now consider the inverse image $R \subseteq k(Z)$ of S under $\psi^*\colon k(Z) \to k(V)$. The ring R is a valuation ring of $k(Z)/k$, because its image S is a valuation ring of $k(V)/k$. By the hypothesis of the theorem there is a point $P \in Z$ such that local ring $\mathcal{O}_{P,Z}$ of Z at P is contained in R. We then have

$$k[Z] \subseteq \mathcal{O}_{P,Z} \subseteq R \xrightarrow{\psi^*} S \longrightarrow k$$

By construction, S contains $k[Y] \subseteq k(V)$, and it contains the injective image of $k[Z]$ under the map above. It follows that S contains the surjective image of $k[Z \times Y] \simeq k[Z] \otimes k[Y]$ in $k[V]$ under the morphism dual to the inclusion $V \subseteq Z \times Y$, and therefore S contains $k[V] \subseteq k(V)$. The intersection of $\ker \Phi$ with $k[V]$ is a maximal ideal of $k[V]$ corresponding to a point in V. This point must be (P, Q); in fact it suffices to show the second coordinate is Q, and this is clear: the map $\Phi\colon S \to k$ is an extension of $\phi\colon k[Y] \to k$, and for any $Q' \neq Q$ we can find a function in $k[Y]$ that vanishes at Q but not at Q' (since $k = \bar{k}$). $\square$

The lemma used in the proof above is a standard result in commutative algebra that we won't prove here.

Lemma 16.29. *Let A be an integral domain contained in a field K and let $\phi\colon A \to k$ be a homomorphism to an algebraically closed field k. Then there exists a valuation ring B of K containing A and a homomorphism $\Phi\colon B \to k$ that extends ϕ. The kernel of Φ is then the maximal ideal of B and k is its residue field.*

Proof. Apply Propositions 5.21 and 5.23 of [1]. □

It will follow easily from Theorem 16.28 that all projective varieties are complete once we prove two lemmas. The first is a technical result that allows us to restrict the residue field of the valuation ring R that appears in the hypothesis of the thoerem.

Lemma 16.30. *Let R be a valuation ring of an extension F/k of an algebraically closed field k. Then there is a valuation ring $R' \subseteq R$ of F/k with residue field isomorphic to k.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R and let $K = R/\mathfrak{m}$ be its residue field. We may view k as a subfield of K, since the map $k \subseteq R \to R/\mathfrak{m} = K$ is a ring homomorphism of fields. So k is an integral domain contained in K, and the identity map $\phi\colon k \to k$ is a homomorphism to an algebraically closed field. By Lemma 16.29, there is a valuation ring S of K/k whose residue field is k. The map $k \subseteq S \to k$ is then the identity map.

The preimage of $R' = \Psi^{-1}(S) \subseteq R$ under the quotient map $\Psi\colon R \to K$ is a subring of R, and the kernel of the map $R' \to S \to k$ is a maximal ideal $\mathfrak{m}'$ (since k is a field), and $\mathfrak{m}'$ contains $\mathfrak{m} = \Phi^{-1}(0)$. We claim that R' is a valuation ring of F/k. It is clear that R' contains k, we just need to show that it is a valuation ring of F.

So let $x \in F$. If $x \notin R$ then $1/x \in \mathfrak{m} \subseteq \mathfrak{m}' \subseteq R'$. If $x \notin R$ but $x \notin R'$, then $x \notin \mathfrak{m}'$ and therefore $x \notin \mathfrak{m}$, implying that $1/x \in R$, since R is a valuation ring. The image of x in K under the quotient map $R \to K$ does not lie in S, since $x \notin R'$, so the image of $1/x$ in K must lies in S, since S is a valuation ring of K/k. Therefore $1/x \in R'$. Thus for every $x \in F$ either x or $1/x$ lies in R'. So R' is a valuation ring of F, and $R'/\mathfrak{m}' \simeq k$. □

Corollary 16.31. *If X is a variety such that for every subvariety $Z \subseteq X$ and valuation ring R of $k(Z)/k$ with residue field k there is a point $P \in Z$ such that $\mathcal{O}_{P,Z} \subseteq R$, then X is complete.*

The next lemma is almost trivial, but it is the essential reason why projective varieties are complete (in contrast to affine varieties), so we consider it separately.

Lemma 16.32. *Let R be a valuation ring of F. For any $x_0, \ldots, x_n \in F^\times$ there exists $\lambda \in F^\times$ such that $\lambda x_0, \ldots, \lambda x_n \in R$ and at least one λx_i is a unit in R.*

Proof. We proceed by induction. For $n = 0$ we may take $\lambda = 1/x_0$ so that $\lambda x_0 = 1 \in R^\times$. We now assume $\lambda x_0, \ldots, \lambda x_{n-1} \in R$ with $\lambda x_i \in R^\times$ for some $i < n$. If $\lambda x_n \in R$ then we are done, and otherwise $1/(\lambda x_n) \in R$ and we let $\lambda' = 1/x_n$. Then $\lambda' x_j = x_i/x_n = \lambda x_j/(\lambda x_n)$ lies in R for $j < n$, and $\lambda' x_n = 1 \in R^\times$. □

Theorem 16.33. *All projective varieties are complete.*

Proof. By Lemma 16.13, it is enough to show that $\mathbb{P}^n$ is complete. To do this we apply Corollary 16.31. Let Z be a variety in $\mathbb{P}^n$ and let R be a valuation ring of $k(Z)/k$ with residue field k. We will construct a point $P \in Z$ for which $\mathcal{O}_P \subseteq R$.

Let $y_0, \ldots, y_n$ be homogeneous coordinates for $\mathbb{P}^n$ and let $z_0, \ldots, z_n$ denote their images in $k(Z)$. Recall that elements of $k(Z)$ can be represented as rational functions whose numerator and denominator are homogeneous polynomials of the same degree; these correspond to homegenizations of elements of $k(Z_i)$ with respect to y_i, where $Z_i = Z \cap Z_i$ is a nonempty affine part of Z.

By Lemma 16.32 there exists $\lambda \in k(Z)^\times$ such that $\lambda z_0, \ldots, \lambda z_n \in R$ with at least one $\lambda z_i \in R^\times$. Let $\phi\colon R \to k$ be the quotient map from R to its residue field, and let P be the projective point $(\phi(\lambda z_0) : \phi(\lambda z_1) : \ldots : \phi(\lambda z_n))$, where we note that at least one $\phi(\lambda z_i)$ is nonzero. The point P lies in Z, since for any homogeneous $f \in I(Z)$ of degree d we have

$$f(\lambda z_0, \ldots, \lambda z_n) = \lambda^d f(z_0, \ldots, z_n) = 0$$

as an element of $k(Z)$, and therefore

$$0 = \phi(0) = \phi(f(\lambda z_0, \ldots, \lambda z_n)) = f(\phi(\lambda z_0), \ldots, \phi(\lambda z_n)) = f(P).$$

Any element of the local ring $\mathcal{O}_P$ can be written as g/h with $h(P) \neq 0$, and we can write g and h as homogeneous polynomials in $\lambda z_0, \ldots, \lambda z_n$ that lie in R (since the λz_i generate $k(Z)$ as a k-algebra). We then have

$$\phi(h(\lambda z_0, \ldots, \lambda z_n)) = h(\phi(\lambda z_0), \ldots, \phi(\lambda z_n)) = h(P) \neq 0,$$

so $h \notin \ker \phi$, and therefore $h \in R^\times$, so $g/h \in R$. Thus $\mathcal{O}_P \subseteq R$, as desired. $\square$

References

[1] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebara*, Addison-Wesley, 1969.

[2] D. Bump, *Algebraic geometry*, World Scientific, 1998.