18.600: Lecture 25

Conditional expectation

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Outline

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Conditional probability distributions

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Interpretation and examples

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- Marginal law of X is weighted average of conditional laws.

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- In continuum setting we had $f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}$. So $E[X|Y=y]=\int_{-\infty}^{\infty} x\frac{f(x,y)}{f_Y(y)}dx$

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- Above fact breaks variance into two parts, corresponding to these two stages.

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- Can we check the formula $\mathrm{Var}(Z) = \mathrm{Var}(E[Z|X]) + E[\mathrm{Var}(Z|X)]$ in this case?

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- But what if we allow non-constant predictors? What if the predictor is allowed to depend on the value of a random variable X that we can observe directly?
- Let $g(x)$ be such a function. Then $E[(y - g(X))^2]$ is minimized when $g(X) = E[Y|X]$.

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- $2 + 3 \cdot 2/50$

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