18.600: Lecture 17

Continuous random variables

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Outline

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Expectation and variance of continuous random variables

Uniform random variable on $[0, 1]$

Uniform random variable on $[\alpha, \beta]$

Measurable sets and a famous paradox

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- Probability of any single point is zero.
- Define **cumulative distribution function** $F(a) = F_X(a) := P\{X < a\} = P\{X \leq a\} = \int_{-\infty}^{a} f(x)dx$.

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- We say that X is **uniformly distributed on** $[0,2]$.

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- This formula is often useful for calculations.

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- $\mathrm{Var} E[X^2] - E[X]^2 = 1/3 - 1/4 = 1/12$.

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- Then expectation linearity gives $E[X] = (\beta - \alpha)E[Y] + \alpha = (1/2)(\beta - \alpha) + \alpha = \frac{\alpha+\beta}{2}$.

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- What is $E[X]$?
- Intuitively, we'd guess the midpoint $\frac{\alpha+\beta}{2}$.
- What's the cleanest way to prove this?
- One approach: let Y be uniform on $[0,1]$ and try to show that $X = (\beta-\alpha)Y + \alpha$ is uniform on $[\alpha,\beta]$.
- Then expectation linearity gives $E[X] = (\beta-\alpha)E[Y] + \alpha = (1/2)(\beta-\alpha) + \alpha = \frac{\alpha+\beta}{2}$.
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- Answer: $\mathrm{Var}[X] = \mathrm{Var}[(\beta-\alpha)Y + \alpha] = \mathrm{Var}[(\beta-\alpha)Y] = (\beta-\alpha)^2\mathrm{Var}[Y] = (\beta-\alpha)^2/12$.

Outline

Continuous random variables

Expectation and variance of continuous random variables

Uniform random variable on $[0, 1]$

Uniform random variable on $[\alpha, \beta]$

Measurable sets and a famous paradox

Outline

Uniform measure: is probability defined for all subsets?

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- What if B is the set of all rational numbers?
- How do we mathematically define the volume of an arbitrary set B?

Idea behind parodox

- **Hypothetical:** Consider the interval $[0, 1)$ with the two endpoints glued together (so it looks like a circle). *What if* we could partition $[0, 1)$ into a countably infinite collection of disjoint sets that all looked the same (up to a rotation of the circle) and thus had to have the same probability?

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- **Related problem:** *if* (in a non-atomic world, where mass was infinitely divisible) you could cut a cake into countably infinitely many pieces all of the same weight, how much would each piece weigh?
- **Question:** Is it really possible to partition $[0, 1)$ into countably many identical (up to rotation) pieces?

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- Now observe that every number in $\{0, 1, 2, \ldots, 99\}$ lies in exactly one of the ten S_j sets we have defined.
- On next slide, we're going to do something similar with $[0, 1)$ in place of $\{0, 1, 2, \ldots, 99\}$ and the **rational numbers in** $[0, 1)$ in place of $\{0, 10, 20, \ldots, 90\}$.

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- Thus $[0, 1) = \cup \tau_r(A)$ as r ranges over rationals in $[0, 1)$.
- If $P(A) = 0$, then $P(S) = \sum_r P(\tau_r(A)) = 0$. If $P(A) > 0$ then $P(S) = \sum_r P(\tau_r(A)) = \infty$. Contradicts $P(S) = 1$ axiom.

Three ways to get around this

- 1. **Re-examine axioms of mathematics:** the very *existence* of a set A with one element from each equivalence class is consequence of so-called **axiom of choice**. Removing that axiom makes paradox goes away, since one can just suppose (pretend?) these kinds of sets don't exist.

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- In practice, sets we care about (e.g., countable unions of points and intervals) tend to be measurable.

Perspective

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- Riemann integration is a mathematically rigorous theory. It's just not as robust as Lebesgue integration.

18.600 Probability and Random Variables
Fall 2019