18.600: Lecture 15

Lectures 1-14 Review

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Outline

Counting tricks and basic principles of probability

Discrete random variables

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Selected counting tricks

- Break "choosing one of the items to be counted" into a sequence of stages so that one always has the same number of choices to make at each stage. Then the total count becomes a product of number of choices available at each stage.

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- How many sequences $a_1, \dots, a_k$ of non-negative integers satisfy $a_1 + a_2 + \dots + a_k = n$?
- Answer: $\binom{n+k-1}{n}$. Represent partition by $k-1$ bars and n stars, e.g., as $** | **|| ****|*$.

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- Countable additivity: $P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ if $E_i \cap E_j = \emptyset$ for each pair i and j.

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- The notation $\sum_{i_1 < i_2 < \dots < i_r}$ means a sum over all of the $\binom{n}{r}$ subsets of size r of the set $\{1, 2, \dots, n\}$.

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- $P(\cup_{i=1}^n E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots \pm \frac{1}{n!}$
- $1 - P(\cup_{i=1}^n E_i) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots \pm \frac{1}{n!} \approx 1/e \approx .36788$

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- Useful when we think about multi-step experiments.
- For example, let E_i be event ith person gets own hat in the n-hat shuffle problem.

Dividing probability into two cases

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- In words: want to know the probability of E. There are two scenarios F and F^c. If I know the probabilities of the two scenarios and the probability of E conditioned on each scenario, I can work out the probability of E.

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- So $P(A|B)$ is $\frac{P(B|A)}{P(B)}$ times $P(A)$.
- Ratio $\frac{P(B|A)}{P(B)}$ determines "how compelling new evidence is".

$P(\cdot|F)$ is a probability measure

- We can check the probability axioms: $0 \leq P(E|F) \leq 1$, $P(S|F) = 1$, and $P(\cup E_i) = \sum P(E_i|F)$, if i ranges over a countable set and the E_i are disjoint.

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- $P(\cdot)$ is the *prior* probability measure and $P(\cdot|F)$ is the *posterior* measure (revised after discovering that F occurs).

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- Equivalent statement: $P(E|F) = P(E)$. Also equivalent: $P(F|E) = P(F)$.

Independence of multiple events

- Say $E_1 \dots E_n$ are independent if for each $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots n\}$ we have $P(E_{i_1} E_{i_2} \dots E_{i_k}) = P(E_{i_1})P(E_{i_2}) \dots P(E_{i_k})$.

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- Independence implies $P(E_1 E_2 E_3 | E_4 E_5 E_6) = \frac{P(E_1)P(E_2)P(E_3)P(E_4)P(E_5)P(E_6)}{P(E_4)P(E_5)P(E_6)} = P(E_1 E_2 E_3)$, and other similar statements.

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- Does pairwise independence imply independence?
- No. Consider these three events: first coin heads, second coin heads, odd number heads. Pairwise independent, not independent.

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- Write $F(a) = P\{X \leq a\} = \sum_{x \leq a} p(x)$. Call F the **cumulative distribution function**.

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- Example: in n-hat shuffle problem, let E_i be the event ith person gets own hat.
- Then $\sum_{i=1}^{n} 1_{E_i}$ is total number of people who get own hats.

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- Represents weighted average of possible values X can take, each value being weighted by its probability.

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- Agrees with the **SUM OVER POSSIBLE** X **VALUES** definition:

$$E[X] = \sum_{x : p(x) > 0} x p(x).$$

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- This is called the **linearity of expectation**.
- Can extend to more variables $E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$.

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- Very important alternate formula: $\mathrm{Var}[X] = E[X^2] - (E[X])^2$.

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- Proof: $\mathrm{Var}[aX] = E[a^2X^2] - E[aX]^2 = a^2E[X^2] - a^2E[X]^2 = a^2\mathrm{Var}[X]$.

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- If we switch from feet to inches in our "height of randomly chosen person" example, then X, $E[X]$, and $\mathrm{SD}[X]$ each get multiplied by 12, but $\mathrm{Var}[X]$ gets multiplied by 144.

Bernoulli random variables

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- $1 = 1^n = (p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$.
- Number of heads is **binomial random variable with parameters** (n, p).

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- Conclude by additivity of expectation that

$$E[X] = \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n} p = np.$$

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- Can show generally that if $X_1, \ldots, X_n$ independent then
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- This also suggests $E[X] = np = \lambda$ and $\mathrm{Var}[X] = npq \approx \lambda$.

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- The numbers of events occurring in disjoint intervals are independent random variables.
- Probability to see zero events in first t time units is $e^{-\lambda t}$.
- Let T_k be time elapsed, since the previous event, until the kth event occurs. Then the T_k are independent random variables, each of which is exponential with parameter λ.

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