18.435/2.111 Homework # 4 Solutions

Problem 1: In the teleportation protocol, show that the probability distribution for the values of the two qubits that Alice sends to Bob is independent of the state ψ of the qubit being transmitted.

Solution to 1:

There are many ways of doing this problem. Writing everything out explicitly gives a straightforward, and not too complicated proof. This is done on page 108 of Nielsen and Chuang (something I didn't realize when I assigned the problem). Here's another proof, using properties of Pauli matrices:

Alice measures $\frac{1}{\sqrt{2}}\,|\,\psi\rangle \otimes (|\,01\rangle - |\,10\rangle)$ in the Bell basis. We want to show that the probability of obtaining each of the four Bell states is 1/4. The Bell basis Alice measures in consists of

$$|\,\psi_{EPR}\rangle = \frac{1}{\sqrt{2}}(|\,01\rangle - |\,10\rangle)$$

and $\sigma_b^{(2)}\,|\,\psi_{EPR}\rangle$ where $b = x, y, z$ and the superscript 2 means that the Pauli matrix is applied to the second qubit. So we want to show that the projection

$${}_{12}\langle\psi_{EPR}\,|\,\sigma_b^{(2)\dagger}(|\,\psi\rangle_1 \otimes |\,\psi_{EPR}\rangle_{23})$$

is independent of b. (The subscripts on $\langle\,|$ and $|\,\rangle$ indicate which qubits these states describe.) This can be seen by realizing that the above measurement gives the same result as projecting the state $\sigma_b^{(2)\dagger}(|\,\psi\rangle_1 \otimes |\,\psi_{EPR}\rangle_{23})$ onto ${}_{12}\langle\psi_{EPR}\,|$. But because applying the same change of basis to both qubits in ψ_{EPR} gives ψ_{EPR} back, we have

$$\sigma_b^{(2)\dagger}(|\,\psi\rangle_1 \otimes |\,\psi_{EPR}\rangle_{23}) = \sigma_b^{(3)}(|\,\psi\rangle_1 \otimes |\,\psi_{EPR}\rangle_{23})$$

and the probability that Alice obtains ${}_{12}\langle\psi_{EPR}\,|$ when she measures this state in the Bell basis cannot be changed if Bob applies $\sigma_b^{(3)}$ to his qubit. Thus, all the probabilities must be equal.

Solution 2:

Alice and Bob share four qubits in the state

$$\frac{1}{2}\,(|\,0000\rangle + |\,0101\rangle + |\,1010\rangle - |\,1111\rangle)$$

This state is just $S(|\,\psi_{EPR}\rangle \otimes |\,\psi_{EPR}\rangle)$, where S is a controlled σ_z. If Alice takes a two-qubit state $|\,\phi\rangle$ and performs the regular teleportation protocol on her two qubits, Bob ends up with

$$S(\sigma_b^{(1)} \otimes \sigma_b^{(2)})\,|\,\phi\rangle\,,$$

where σ_b is either the identity or one of the four Pauli matrices. He now needs to convert this to $S\phi$. It is easy to see that $\sigma_z^{(i)}$ commutes with S where $i = 1, 2$, and that

$$\begin{aligned} S\sigma_x^{(1)} &= \sigma_z^{(2)}\sigma_x^{(1)}S \\ S\sigma_x^{(2)} &= \sigma_z^{(1)}\sigma_x^{(2)}S \end{aligned}$$

From these, and the relation $\sigma_y = i\sigma_x\sigma_z$, we can (assuming no calculation mistakes on my part) derive the following table.

Bob's correction in regular teleportation	Bob's correction teleporting through S
id	id
$\sigma_x^{(2)}$	$\sigma_z^{(1)} \otimes \sigma_x^{(2)}$
$\sigma_y^{(2)}$	$\sigma_z^{(1)} \otimes \sigma_y^{(2)}$
$\sigma_z^{(2)}$	$\sigma_z^{(2)}$
$\sigma_x^{(1)}$	$\sigma_x^{(1)} \otimes \sigma_z^{(2)}$
$\sigma_x^{(1)} \otimes \sigma_x^{(2)}$	$\sigma_y^{(1)} \otimes \sigma_y^{(2)}$
$\sigma_x^{(1)} \otimes \sigma_y^{(2)}$	$\sigma_y^{(1)} \otimes \sigma_x^{(2)}$
$\sigma_x^{(1)} \otimes \sigma_z^{(2)}$	$\sigma_x^{(1)}$
$\sigma_y^{(1)}$	$\sigma_y^{(1)} \otimes \sigma_z^{(2)}$
$\sigma_y^{(1)} \otimes \sigma_x^{(2)}$	$\sigma_x^{(1)} \otimes \sigma_y^{(2)}$
$\sigma_y^{(1)} \otimes \sigma_y^{(2)}$	$\sigma_x^{(1)} \otimes \sigma_x^{(2)}$
$\sigma_y^{(1)} \otimes \sigma_z^{(2)}$	$\sigma_y^{(1)}$
$\sigma_z^{(1)}$	$\sigma_z^{(1)}$
$\sigma_z^{(1)} \otimes \sigma_x^{(2)}$	$\sigma_x^{(2)}$
$\sigma_z^{(1)} \otimes \sigma_y^{(2)}$	$\sigma_y^{(2)}$
$\sigma_z^{(1)} \otimes \sigma_z^{(2)}$	$\sigma_z^{(1)} \otimes \sigma_z^{(2)}$

The mapping between Alice's measurement and Bob's correction is now straightforward to compute, given the map between Alice's mesurement and Bob's correction in regular teleportation.

Problem 3:
If Alice and Bob share a set of qutrits in the state

$$\frac{1}{\sqrt{3}}(|\,00\rangle + |\,11\rangle + |\,22\rangle),$$

show that Alice can do superdense coding by applying $R^a T^b$ to this state, for $0 \le a \le 2$ and $0 \le b \le 2$, where

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

where $\omega = e^{2\pi i/3}$. Note that I left out the definition of ω in the problem set, but most people figured it out.
Solution to 3: We need to show that

$$\langle EPR_3\,|\,(T^{\dagger^{b'}} R^{\dagger^{a'}} \otimes I)(R^a T^b \otimes I)\,|\,EPR_3\rangle = \delta_{a-a'}\delta_{b-b'}$$

where δ is the Kronecker δ function. This will show that the nine states Alice produces are an orthonormal basis, so when she sends her qutrit to Bob, he can distinguish all nine states using a von Neumann measurement. We can use the fact that $R^3 = T^3 = I$ and that $TR = \omega RT$ to simplify

$$T^{\dagger b'} R^{\dagger a'} R^a T^b = \omega^{-b'(a-a')} R^{a-a'} T^{b-b'}.$$

This means we merely need to show that

$$\langle EPR_3 \,|\, R^a T^b \otimes I \,|\, EPR_3 \rangle = \delta_a \delta_b$$

for $0 \le a, b \le 2$. If $b \neq 0$, then $R^a T^b \otimes I \,|\, EPR_3\rangle$ is a superposition of basis states of the form $|\, ij\rangle$ for $i \neq j$, and so has inner product 0 with $|\, EPR_3\rangle$. If $b = 0$, then

$$R^a \,|\, EPR_3\rangle = \frac{1}{\sqrt{3}}(|\, 00\rangle + \omega^a \,|\, 11\rangle + \omega^{2a} \,|\, 22\rangle)$$

and the inner product of this with $|\, EPR_3\rangle$ is $(1 + \omega^a + \omega^{2a})/3$, which if we choose $\omega = e^{2\pi i/3}$ is 1 if $a = 0$, and 0 if $a = 1, 2$.

Solution for 4: Alice and Cathy share a Bell state, which can be written as

$$\sigma_1^{(C)} \,|\, \psi_{EPR}\rangle_{AC},$$

where σ_1 is either one of the three Pauli matrices or the identity. The (C) represents that it is applied to Cathy's qubit [note that this really should be written $id^{(B)} \otimes \sigma_1^{(C)}$, but we are leaving out implied identity matrices, as this notation gets cumbersome very quickly]. Alice and Cathy don't know what σ_1 is, but they know that it is the same as the σ_1 in the state Bob and David share, which is

$$\sigma_1^{(D)} \,|\, \psi_{EPR}\rangle_{BD}.$$

Now, if Alice uses

$$\sigma_1^{C} \,|\, \psi_{EPR}\rangle_{AC}$$

to teleport her qubit of $|\, \psi_{EPR}\rangle_{AB}$ to Cathy, what happens is that Cathy and Bob now hold $\sigma_1^C \sigma_2^C \,|\, \psi_{EPR}\rangle_{CB}$, where Cathy knows what σ_2^C is (because this depends on the results of Alice's measurement) but not σ_1. Now, Bob uses

$$\sigma_1^{D} \,|\, \psi_{EPR}\rangle_{BD}$$

to teleport his qubit of $\sigma_1^C \sigma_2^C \,|\, \psi_{EPR}\rangle_{CB}$ to David. Now, Cathy and David share

$$\sigma_1^C \sigma_2^C \otimes \sigma_1^D \sigma_3^D \,|\, \psi_{EPR}\rangle_{CD} = \pm \sigma_2^C \sigma_1^C \otimes \sigma_3^D \sigma_1^D \,|\, \psi_{EPR}\rangle_{CD},$$

where we can interchange the two pairs of Pauli matrices because any two Pauli matrices either commute or anticommute. But since Cathy and David know σ_2 and σ_3, they can undo them, leaving

$$\pm \sigma_1^C \otimes \sigma_1^D \,|\, \psi_{EPR}\rangle_{CD}.$$

The ± 1 phase factor does not change the quantum state, and since the state $|\,\psi_{EPR}\rangle_{CD}$ is invariant when the same basis transformation is applied to both of its qubits, Cathy and David now share

$$\pm\,|\,\psi_{EPR}\rangle_{CD}\,,$$

which is what we wanted.

Problem 5. It's late, and problem 5 is not only extra credit, but also quite tricky, so I'll post the solution to it later.