18.404/6.840 Lecture 9

Last time:

- A_{TM} is undecidable
- The diagonalization method
- $\overline{A_{\mathrm{TM}}}$ is T-unrecognizable
- The Reducibility Method, preview

Today: (Sipser §5.1, §5.3)

- The Reducibility Method for proving undecidability and T-unrecognizability.
- General reducibility
- Mapping reducibility

The Reducibility Method

If we know that some problem (say A_{TM}) is undecidable,
we can use that to show other problems are undecidable.

Defn: $HALT_{\text{TM}} = \{\langle M, w\rangle |\ M \text{ halts on input } w\}$

Recall Theorem: $HALT_{\text{TM}}$ is undecidable
Proof by contradiction, showing that A_{TM} is reducible to $HALT_{\text{TM}}$:

Assume that $HALT_{\text{TM}}$ is decidable and show that A_{TM} is decidable (false!).
Let TM R decide $HALT_{\text{TM}}$.
Construct TM S deciding A_{TM}.

$S =$ "On input $\langle M, w\rangle$

1. Use R to test if M on w halts. If not, *reject*.
2. Simulate M on w until it halts (as guaranteed by R).
3. If M has accepted then *accept*.
   If M has rejected then *reject*.

TM S decides A_{TM}, a contradiction. Therefore $HALT_{\text{TM}}$ is undecidable.

Reducibility – Concept

If we have two languages (or problems) A and B, then
A is reducible to B means that we can use B to solve A.

Example 1: Measuring the area of a rectangle
is reducible to measuring the lengths of its sides.

Example 2: We showed that A_{NFA} is reducible to A_{DFA} .

Example 3: From Pset 2, *PUSHER* is reducible to E_{CFG} .
(Idea- Convert push states to accept states.)

If A is reducible to B then solving B gives a solution to A.
- then B is easy $\rightarrow$ A is easy.
- then A is hard $\rightarrow$ B is hard.

this is the form we will use

Check-in 9.1

Is Biology reducible to Physics?

(a) Yes, all aspects of the physical world may be explained in terms of Physics, at least in principle.

(b) No, some things in the world, maybe life, the brain, or consciousness, are beyond the realm pf Physics.

(c) I'm on the fence on this question!

Check-in 9.1

E_{TM} is undecidable

Let $E_{\mathrm{TM}} = \{\langle M\rangle \mid M$ is a TM and $L(M) = \emptyset\}$

Theorem: E_{TM} is undecidable

Proof by contradiction. Show that A_{TM} is reducible to E_{TM}.

Assume that E_{TM} is decidable and show that A_{TM} is decidable (false!).
Let TM R decide E_{TM}.
Construct TM S deciding A_{TM}.

$S =$ "On input $\langle M, w\rangle$

1. Transform M to new TM $M_w =$ "On input x
    1. If $x \neq w$, *reject*.
    2. else run M on w
    3. *Accept* if M accepts."
2. Use R to test whether $L(M_w) = \emptyset$
3. If YES [so M rejects w] then *reject*.
   If NO [so M accepts w] then *accept*.

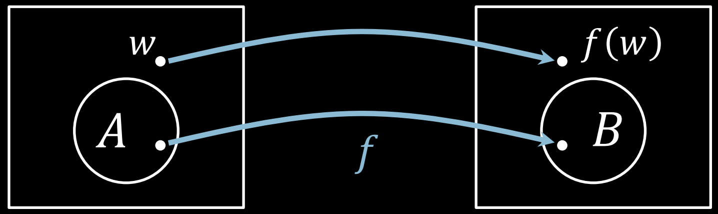
M_w works like M except that it always rejects strings x where $x \neq w$.

So $L(M_w) = \begin{cases} \{w\} & \text{if } M \text{ accepts } w \\ \emptyset & \text{if } M \text{ rejects } w \end{cases}$

Mapping Reducibility

Defn: Function $f: \Sigma^* \to \Sigma^*$ is computable if there is a TM F where F on input w halts with $f(w)$ on its tape, for all strings w.

Defn: A is mapping-reducible to B ($A \leq_{\mathrm{m}} B$) if there is a computable function f where $w \in A$ iff $f(w) \in B$.



Example: $A_{\mathrm{TM}} \leq_{\mathrm{m}} \overline{E_{\mathrm{TM}}}$

The computable reduction function f is $f(\langle M, w\rangle) = \langle M_w\rangle$

Because $\langle M, w\rangle \in A_{\mathrm{TM}}$ iff $\langle M_w\rangle \in \overline{E_{\mathrm{TM}}}$

(M accepts w iff $L(\langle M_w\rangle) \neq \emptyset$)

Recall TM M_w = "On input x

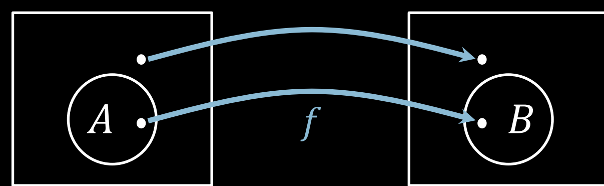
1. If $x \neq w$, *reject*.
2. else run M on w
3. *Accept* if M accepts."

Mapping Reductions - properties

Theorem: If $A \leq_{\mathrm{m}} B$ and B is decidable then so is A

Proof: Say TM R decides B.

Construct TM S deciding A:



$S =$ "On input w

1. Compute $f(w)$
2. Run R on $f(w)$ to test if $f(w) \in B$
3. If R halts then output same result."

Corollary: If $A \leq_{\mathrm{m}} B$ and A is undecidable then so is B

Theorem: If $A \leq_{\mathrm{m}} B$ and B is T-recognizable then so is A

Proof: Same as above.

Corollary: If $A \leq_{\mathrm{m}} B$ and A is T-unrecognizable then so is B

Check-in 9.2

Suppose $A \leq_{\mathrm{m}} B$.

What can we conclude?

Check all that apply.

(a) $B \leq_{\mathrm{m}} A$

(b) $\overline{A} \leq_{\mathrm{m}} \overline{B}$

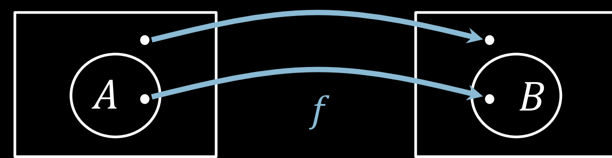
(c) None of the above

Check-in 9.2

Mapping vs General Reducibility

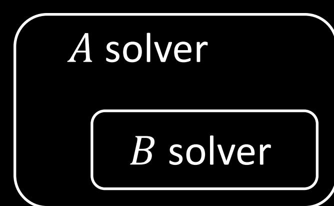
Mapping Reducibility of A to B: Translate A-questions to B-questions.

- A special type of reducibility
- Useful to prove T-unrecognizability



(General) Reducibility of A to B: Use B solver to solve A.

- May be conceptually simpler
- Useful to prove undecidability



Noteworthy difference:

- A is reducible to $\overline{A}$
- A may not be mapping reducible to $\overline{A}$.

  For example $\overline{A_{\mathrm{TM}}} \not\leq_{\mathrm{m}} A_{\mathrm{TM}}$

Check-in 9.3

We showed that if $A \leq_{\mathrm{m}} B$ and B is T-recognizable then so is A.

Is the same true if we use general reducibility instead of mapping reducibility?

(a) Yes

(b) No

Check-in 9.3

Reducibility – Templates

To prove B is undecidable:
- Show undecidable A is reducible to B. (often A is A_{TM})
- Template: Assume TM R decides B.
  Construct TM S deciding A. Contradiction.

To prove B is T-unrecognizable:
- Show T-unrecognizable A is mapping reducible to B. (often A is $\overline{A_{\mathrm{TM}}}$)
- Template: give reduction function f.

E_{TM} is T-unrecognizable

Recall $E_{\mathrm{TM}} = \{\langle M \rangle |\ M$ is a TM and $L(M) = \emptyset\ \}$

Theorem: E_{TM} is T-unrecognizable

Proof: Show $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} E_{\mathrm{TM}}$

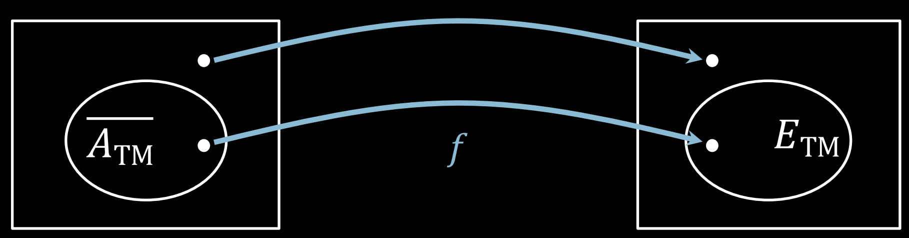
Reduction function: $f(\langle M, w \rangle) = \langle M_w \rangle$

Recall TM $M_w =$ "On input x

1. If $x \neq w$, *reject*.
2. else run M on w
3. *Accept* if M accepts."

Explanation: $\langle M, w \rangle \in \overline{A_{\mathrm{TM}}}$ iff $\langle M_w \rangle \in E_{\mathrm{TM}}$

M rejects w iff $L(\langle M_w \rangle) = \emptyset$



EQ_{TM} and $\overline{EQ_{\mathrm{TM}}}$ are T-unrecognizable

$EQ_{\mathrm{TM}} = \{\langle M_1, M_2\rangle |\ M_1$ and M_2 are TMs and $L(M_1) = L(M_2)\ \}$

Theorem: Both EQ_{TM} and $\overline{EQ_{\mathrm{TM}}}$ are T-unrecognizable

Proof: (1) $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$

(2) $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} \overline{EQ_{\mathrm{TM}}}$

For any w let T_w = "On input x

1. Ignore x.
2. Simulate M on w."

T_w acts on all inputs the way M acts on w.

(1) Here we give f which maps $\overline{A_{\mathrm{TM}}}$ problems (of the form $\langle M, w\rangle$) to EQ_{TM} problems (of the form $\langle T_1, T_2\rangle$).

$f(\langle M, w\rangle) = \langle T_w, T_{\text{reject}}\rangle$ $\quad T_{\text{reject}}$ is a TM that always rejects.

(2) Similarly $f(\langle M, w\rangle) = \langle T_w, T_{\text{accept}}\ \rangle$ $\quad T_{\text{accept}}$ always accepts.

Reducibility terminology

Why do we use the term “reduce”?

When we reduce A to B, we show how to solve A by using B
and conclude that A is no harder than B. (suggests the $\leq_{\mathrm{m}}$ notation)

Possibility 1: We bring A’s difficulty down to B’s difficulty.

Possibility 2: We bring B’s difficulty up to A’s difficulty.

Quick review of today

1. Introduced The Reducibility Method to prove undecidability and T-unrecognizability.
2. Defined mapping reducibility as a type of reducibility.
3. E_{TM} is undecidable.
4. E_{TM} is T-unrecognizable.
5. EQ_{TM} and $\overline{EQ_{\mathrm{TM}}}$ are T-unrecognizable.