18.404/6.840 Lecture 8

Last time:

- Decision procedures for automata and grammars

  A_{DFA} , A_{NFA} , E_{DFA} , EQ_{DFA} , A_{CFG} , E_{CFG} are decidable

  A_{TM} is T-recognizable

Today: (Sipser §4.2)

- A_{TM} is undecidable
- The diagonalization method
- $\overline{A_{\mathrm{TM}}}$ is T-unrecognizable
- The reducibility method
- Other undecidable languages

Recall: Acceptance Problem for TMs

Let $A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M$ is a TM and M accepts $w\}$

Today's Theorem: A_{TM} is not decidable

Proof uses the diagonalization method,
so we will introduce that first.

The Size of Infinity

How to compare the relative sizes of infinite sets?

Cantor (~1890s) had the following idea.

Defn: Say that set A and B have the same size if there is a one-to-one and onto function $f: A \to B$

one-to-one: $x \neq y \to f(x) \neq f(y)$ "injective"

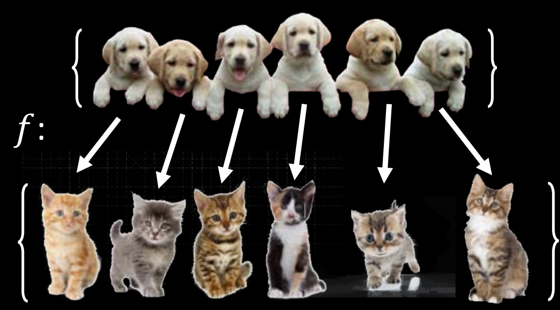
onto: Range $(f) = B$ "surjective"

We call such an f a 1-1 correspondence

Informally, two sets have the same size if we can pair up their members.

This definition works for finite sets.

Apply it to infinite sets too.



Countable Sets

Let $\mathbb{N} = \{1,2,3,\ldots\}$ and let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Show $\mathbb{N}$ and $\mathbb{Z}$ have the same size

n	$f(n)$
$\mathbb{N}$	$\mathbb{Z}$

Let $\mathbb{Q}^+ = \{\, {}^{m}/_{n} \mid m, n \in \mathbb{N}\}$

Show $\mathbb{N}$ and $\mathbb{Q}^+$ have the same size

$\mathbb{Q}^+$	1	2	3	4	…
1	1/1	1/2	1/3	1/4	
2	2/1	2/2	2/3	2/4	…
3	3/1	3/2	3/3	3/4	
4	4/1	4/2	4/3	4/4	
⋮		⋮			

n	$f(n)$
$\mathbb{N}$	$\mathbb{Q}^+$

Defn: A set is <u>countable</u> if it is finite or it has the same size as $\mathbb{N}$.

Both $\mathbb{Z}$ and $\mathbb{Q}^+$ are countable.

$\mathbb{R}$ is Uncountable – Diagonalization

Let $\mathbb{R}$ = all real numbers (expressible by infinite decimal expansion)

Theorem: $\mathbb{R}$ is uncountable

Proof by contradiction via diagonalization: Assume $\mathbb{R}$ is countable

So there is a 1-1 correspondence $f: \mathbb{N} \rightarrow \mathbb{R}$

n	$f(n)$
1	
2	
3	
4	
5	
6	
7	
$\vdots$	

Diagonalization

Demonstrate a number $x \in \mathbb{R}$ that is missing from the list.

$$x = 0 .$$

differs from the n^{th} number in the n^{th} digit
so cannot be the n^{th} number for any n.

Hence x is not paired with any n. It is missing from the list.

Therefore f is not a 1-1 correspondence.

$\mathbb{R}$ is Uncountable – Corollaries

Let $\mathcal{L}$ = all languages

Corollary 1: $\mathcal{L}$ is uncountable

Proof: There's a 1-1 correspondence from $\mathcal{L}$ to $\mathbb{R}$ so they are the same size.

Observation: $\Sigma^* = \{\varepsilon, 0,1,00,01,10,11,000, \ldots\}$ is countable.

Let $\mathcal{M}$ = all Turing machines

Observation: $\mathcal{M}$ is countable.

Because $\{\langle M\rangle |\ M \text{ is a TM}\} \subseteq \Sigma^*$.

Corollary 2: Some language is not decidable.

Because there are more languages than TMs.

We will show some specific language A_{TM} is not decidable.

Check-in 8.1

Hilbert's 1st question asked if there is a set of intermediate size between $\mathbb{N}$ and $\mathbb{R}$. Gödel and Cohen showed that we cannot answer this question by using the standard axioms of mathematics. How can we interpret their conclusion?

(a) We need better axioms to describe reality.

(b) Infinite sets have no mathematical reality so Hilbert's 1st question has no answer.

Check-in 8.1

A_{TM} is undecidable

Recall $A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

Theorem: A_{TM} is not decidable

Proof by contradiction: Assume some TM H decides A_{TM}.

So H on $\langle M, w\rangle = \begin{cases} \textit{Accept} & \text{if } M \text{ accepts } w \\ \textit{Reject} & \text{if not} \end{cases}$

Use H to construct TM D

D = "On input $\langle M\rangle$

1. Simulate H on input $\langle M, \langle M\rangle\rangle$
2. *Accept* if H rejects. *Reject* if H accepts."

D accepts $\langle M\rangle$ iff M doesn't accept $\langle M\rangle$.

D accepts $\langle D\rangle$ iff D doesn't accept $\langle D\rangle$.

Contradiction.

Why is this proof a diagonalization?

All TM descriptions:

All TMs ⇩	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	. . .	$\langle D\rangle$
M_1						
M_2						
M_3						
M_4						
⋮						
D						

Check-in 8.2

Recall the Queue Automaton (QA) defined in Pset 2.
It is similar to a PDA except that it is deterministic
and it has a queue instead of a stack.

Let $A_{\mathrm{QA}} = \{\langle B, w\rangle |\ B$ is a QA and B accepts $w\}$

Is A_{QA} decidable?

(a) Yes, because QA are similar to PDA and A_{PDA} is decidable.

(b) No, because “yes” would contradict results we now know.

(c) We don’t have enough information to answer this question.

Check-in 8.2

$\overline{A_{\text{TM}}}$ is T-unrecognizable

Theorem: If A and $\overline{A}$ are T-recognizable then A is decidable

Proof: Let TM M_1 and M_2 recognize A and $\overline{A}$.

Construct TM T deciding A.

$T =$ "On input w

1. Run M_1 and M_2 on w in parallel until one accepts.
2. If M_1 accepts then *accept*.
   If M_2 accepts then *reject*."

Corollary: $\overline{A_{\text{TM}}}$ is T-unrecognizable

Proof: A_{TM} is T-recognizable but also undecidable

Check-in 8.3

From what we've learned, which closure properties can we prove for the class of T-recognizable languages?

Choose all that apply.

(a) Closed under union.

(b) Closed under intersection.

(c) Closed under complement.

(d) Closed under concatenation.

(e) Closed under star.

Check-in 8.3

The Reducibility Method

Use our knowledge that A_{TM} is undecidable to show other problems are undecidable.

Defn: $HALT_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ halts on input } w\}$

Theorem: $HALT_{\mathrm{TM}}$ is undecidable
Proof by contradiction, showing that A_{TM} is reducible to $HALT_{\mathrm{TM}}$:

Assume that $HALT_{\mathrm{TM}}$ is decidable and show that A_{TM} is decidable (false!).
Let TM R decide $HALT_{\mathrm{TM}}$.
Construct TM S deciding A_{TM}.

$S =$ "On input $\langle M, w\rangle$

1. Use R to test if M on w halts. If not, reject.
2. Simulate M on w until it halts (as guaranteed by R).
3. If M has accepted then *accept*.
   If M has rejected then *reject*.

TM S decides A_{TM}, a contradiction. Therefore $HALT_{\mathrm{TM}}$ is undecidable.

Quick review of today

1. Showed that $\mathbb{N}$ and $\mathbb{R}$ are not the same size to introduce the Diagonalization Method.
2. A_{TM} is undecidable.
3. If A and $\overline{A}$ are T-recognizable then A is decidable.
4. $\overline{A_{\mathrm{TM}}}$ is T-unrecognizable.
5. Introduced the Reducibility Method to show that $HALT_{\mathrm{TM}}$ is undecidable.

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