Theory of Computation
18.404/6.840

Michael Sipser

Sample MidTerm Examination Questions

1. (30 points)

   (a) (15 points) Let $A = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^i \mid i \leq j \leq 2i\}$.
   Prove that A is not a context-free language.

   (b) (15 points) Let $B = \{\mathtt{a}^i\mathtt{b}^j \mid i \leq j \leq 2i\}$.
   Give an *unambiguous* context-free grammar generating B. If you have trouble producing an unambiguous grammar for B, you will receive partial credit if you state that you will give an ambiguous grammar instead and give an ambiguous CFG for B.

2. (25 points)

   (a) Let R be a regular expression that has length n. (the *length* of a regular expression is the number of symbols it contains, including members of Σ, operation symbols, and parentheses. For example, the regular expression $(\mathtt{01}^*)\mathtt{0}$ has length 6.) If we convert R to an equivalent NFA N using the procedure we described in class, how many states would N have? Answer this question by giving an approximate bound on the number of states in N.
   Choose the best answer from the possibilities below:

   1. $O(n)$
   2. $O(n \log n)$
   3. $O(n^2)$
   4. $O(n^3)$
   5. $O(4^n)$
   6. Larger than any of the above.

   You do not need to justify your answer. ANSWER: _____.

   (b) Let M be a DFA that has n states. If we convert M to an equivalent regular expression R using the procedure we described, how long would R be?
   Give an approximate bound on the length of R.
   Choose the best answer from the possibilities below:

   1. $O(n)$
   2. $O(n \log n)$
   3. $O(n^2)$
   4. $O(n^3)$
   5. $O(4^n)$
   6. Larger than any of the above.

   You do not need to justify your answer. ANSWER: _____.

   (c) In an *extended* regular expression, we may use the complementation operation ($\neg$) in addition to the three regular operations ($\cup, \circ, *$). For example

   $$\neg(\Sigma^*\mathtt{0001}\Sigma^*) \cup \neg(\Sigma^*\mathtt{1110}\Sigma^*)$$

   is an extended regular expression that describes the collection of all strings that either do not contain the substring `0001` or do not contain the substring `1110`.
   Describe how to modify the conversion procedure from regular expressions to NFAs so that it becomes a conversion procedure from extended regular expressions to NFAs.

(d) Let R be an extended regular expression that has length n. If we convert R to an equivalent NFA N using the procedure you described above, how many states would N have? Here we are seeking an approximate bound on the number of states in N. Choose the best answer from the possibilities below:

1. $O(n)$
2. $O(n \log n)$
3. $O(n^2)$
4. $O(n^3)$
5. $O(4^n)$
6. Larger than any of the above.

You do not need to justify your answer. ANSWER: _____.

3. (25 points) Let $D = \{\langle M \rangle | \ M \text{ is a TM that accepts the input string } \texttt{101}\}$.

   (a) (15 points) Show that D is undecidable.
   (Do not use Rice's theorem. If you don't know Rice's theorem, ignore this comment.)

   (b) (10 points) Show that the complement of D is not Turing-recognizable.

4. (20 points)
   A *2-way pushdown automaton* (2WAY-PDA) is a nondeterministic pushdown automaton that has a single stack and that can move its input head in both directions on the input tape. In addition we assume that a 2WAY-PDA is capable of detecting when its input head is at either end of its input tape. A 2WAY-PDA accepts its input by entering an accept state.

   (a) (5 points) Show that a 2WAY-PDA can recognize the language $\{\texttt{a}^m\texttt{b}^m\texttt{c}^m | \ m \geq 0\}$.

   (b) (15 points) Let $E_{\text{2WAY-PDA}} = \{\langle P \rangle | \ P \text{ is a 2WAY-PDA which recognizes the empty language}\}$.
   Show that $E_{\text{2WAY-PDA}}$ is not decidable.

5. (25 points) Consider the infinite two-dimensional grid, $G = \{(m, n) | \ m \text{ and } n \text{ are integers}\}$. Every point in G has 4 neighbors, North, South, East, and West, obtained by varying m or n by ± 1. Starting at the origin $(0, 0)$, a string of commands `N`, `S`, `E`, `W`, generates a path in G. For example, the string `NESW`, generates a path clockwise around a unit square touching the origin. Say that a path is *closed* if it starts at the origin and ends at the origin.

   Let C be the collection of all strings over $\Sigma = \{\texttt{N}, \texttt{S}, \texttt{E}, \texttt{W}\}$ that generate a closed path.

   (a) Give a clear mathematical description of C as a language.

   (b) Describe in English two CFLs, A and B, such that $C = A \cap B$.
   Give a CFG that generates A.

   (c) Prove that C is not context-free.

6. (25 points) Let $\Sigma = \{\texttt{0}, \texttt{1}\}$. Consider the problem of testing whether a PDA accepts some string of the form $\{w | \ w \in \texttt{0}^*\texttt{1}^*\}$. Is this problem decidable? Prove your answer.

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