FINAL EXAM SOLUTIONS

1. (a) `False`, hierarchy theorem.
(b) `True`, Savitch's theorem.
(c) `True`, PSPACE is closed under complement.
(d) `Open`, stated in lecture.
(e) `True`, follows from definition.
(f) `Open`, implies NP = coNP (considering language $\texttt{0}SAT \cup \texttt{1}\overline{SAT}$); negation implies PSPACE $\neq$ NP.
(g) `Open`, equivalent to NP = coNP.
(h) `Open`, stated in lecture.
(i) `False`, implies PSPACE = EXPSPACE.
(j) `True`, recompute bits of first reduction.
(k) `Open`, equivalent to NP = coNP.
(l) `True`, SAT is decidable.
(m) `False`, implies PSPACE = NL.
(n) `Open`, equivalent to P = NP.
(o) `True`, $PATH \in$ P.
(p) `True`, NL = coNL.

2. First, show that C is in EXPTIME. Here's the algorithm:

"On input $\langle M, w, i, j, \alpha \rangle$:
   1. Run M on w for j steps. If it halts in fewer steps, *reject*.
   2. *Accept* if the ith symbol of the configuration of the jth step is α. Otherwise, *reject*."

To analyze the running time of this algorithm, observe that to simulate one step of M we only need to update M's configuration and the counter which records how long M has been running. Both can be done within $O(j)$ steps (actually much less is possible, but unnecessary here). We run M for at most j steps, so the total running time of this algorithm is $O(j^2)$, and that is exponential in the size of the input, because j represented in binary, so $|j| = \log_2 j$ and thus $j^2 = (2^{|j|})^2 = 2^{2|j|} \leq 2^{2n}$, where n is the length of the entire input.

Second, we show that C is EXPTIME-hard, that is, that every language in EXPTIME is polynomial time reducible to C. Let $A \in$ EXPTIME where M decides A in time 2^{n^k}. Modify M so that when it accepts it first moves its head to the left-hand end of the tape and then enters the accept state q_{accept}. Then the reduction of A to C is the polynomial time computable function f, where $f(w) = \langle M, w, 1, j, q_{\text{accept}} \rangle$ and $j = 2^{n^k}$.

3. First, *SOLITAIRE* $\in$ NP because we can check in polynomial time that a solution works.

Second, show that *3SAT* $\leq_{\text{P}}$ *SOLITAIRE*.
Given ϕ with k variables $x_1, \ldots, x_k$ and l clauses $c_1, \ldots, c_l$, first remove any clauses that contain both x_i and $\overline{x_i}$. These clauses are useless anyway and would mess up the coming construction. Construct the following $l \times k$ game G.

If x_i is in clause c_j put a blue stone in row c_j, column x_i.
If $\overline{x_i}$ is in clause c_j put a red stone in row c_j, column x_i.
(We can make it a square $m \times m$ by repeating a row or adding a blank column as necessary without affecting solvability).
Claim: ϕ is satisfiable iff G has a solution.
($\rightarrow$): Take a satisfying assignment. If x_i is true (false), remove the red (blue) stones from the corresponding column. So, stones corresponding to true literals remain. Since every clause has a true literal, every row has a stone.
($\leftarrow$): Take a game solution. If the red (blue) stones were removed from a column, set the corresponding variable true (false). Every row has a stone remaining, so every clause has a true literal. Therefore ϕ is satisfied.

4. Show that $A_{\mathrm{T}M} \leq_{\mathrm{m}} INP$.
   Assume (to get a contradiction) that TM R decides INP. Construct the following TM S deciding $A_{\mathrm{T}M}$.

   "On input $\langle M, w\rangle$:

   1. Construct the following TM M_1:

   "On input x:

   1. If $x \in EQ_{\mathrm{REX}\uparrow}$, *accept*.

   2. Run M on w.

   3. If M accepts w, *accept*."

   4. Run R on M_1.

   5. If R accepts, *accept*; otherwise, *reject*."

   Observe that if M accepts w, then $L(M_1) = \Sigma^*$, and if M doesn't accept w, then $L(M_1) = EQ_{\mathrm{REX}\uparrow}$. So, $L(M_1) \in \mathrm{P}$ exactly when M accepts w.

5. (a) Obviously *ODD-PARITY* $\in$ L and we know L $\subseteq$ NL. We proved that *PATH* is NP-complete and so every language in NL is log-space reducible to *PATH*.
   Note: Giving a direct reduction from *ODD-PARITY* to *PATH* is possible too.

   (b) If *PATH* $\leq_{\mathrm{L}}$ *ODD-PARITY* then *PATH* $\in$ L and thus NL = L, solving a big open problem.

6. We can assume without loss of generality that our BPP machine makes exactly n^r coin tosses on each branch. Thus the problem of determining the probability of accepting a string reduces to counting how many branches are accepting and comparing this number with $\frac{2}{3}2^{(n^r)}$.

   So given w, we generate all binary strings x of length n^r (we can do this in PSPACE) and simulate M on w using x as the source of randomness. If M accepts, then we increment a count. At the end, we see how many branches have accepted. If that number is more than $\frac{2}{3}2^{(n^r)}$ we accept else we reject. This works because of the definition of what it means for a BPP machine to accept. If $w \in L$ then more than $\frac{2}{3}$ of M's branches must accept. If $w \notin L$ then at most $\frac{1}{3}$ of its branches can accept.

7. (a) No, the prover for $\#SAT$ is not a weak Prover, as far as we know. Calculating the cooeficients of the polynomials seems to require more than polynomial time.

   (b) The class weak-IP = BPP. Clearly, BPP $\subseteq$ weak-IP because the Verifier can simply ignore the Prover. Conversely, weak-IP $\subseteq$ BPP because we can make a BPP machine which simulates both the Verifier and the weak Prover P. If $w \in A$ then P causes the Verifier to accept with high probability and so will the BPP machine. If $w \notin A$ then P causes the Verifier to accept with low probability and so will the BPP machine.