3 Dimensionless groups

A formal justification of the dimensional analysis approach in the previous section comes from Buckingham's Pi Theorem. Consider a physical problem in which the dependent parameter is a function of $n-1$ independent parameters, so that we may express the relationship among the variables in functional form as

$$q_1 = g(q_2, q_3, ..., q_n), \tag{35a}$$

where q_1 is the dependent parameter, and $q_2, ..., q_n$ are the $n-1$ independent parameters. Mathematically, we can rewrite the functional relationship in the equivalent form

$$0 = f(q_1, q_2, ..., q_n). \tag{35b}$$

where $f = q_1 - g(q_2, q_3, ..., q_n)$. For example, for the period of a pendulum we wrote $\tau = \tau(l, g, m)$, but we could just as well have written $f(\tau, l, g, m) = 0$. The Buckingham Pi theorem states that given a relation of the form (35), the n parameters may be grouped into $n-d$ *independent dimensionless* ratios, or dimensionless groups Π_i, expressible in functional form by

$$\Pi_1 = G(\Pi_2, \Pi_3, ..., \Pi_{n-d}), \tag{36a}$$

or, equivalently,

$$0 = F(\Pi_1, \Pi_2, ..., \Pi_{n-d}), \tag{36b}$$

where d is the number of independent dimensions (mass, length, time...). The formal proof can be found in the book *Scaling, Self Similarity and Intermediate Asymptotics* by Barenblatt. The Pi theorem does not predict the functional form of F or G, and this must be determined experimentally. The $n-d$ dimensionless groups Π_i are independent. A dimensionless group Π_i is *not* independent if it can be formed from a product or quotient of other dimensionless groups in the problem.

3.1 The pendulum

To develop an understanding of how to use Buckingham's Pi theorem, let's first apply it to the problem of a swinging pendulum, which we considered in the previous lecture. We argued that the period of the pendulum τ depends on the length l and gravity g. It cannot depend on the mass m since we cannot form a dimensionless parameter including m in our list of physical variables. Thus

$$\tau = \tau(l, g), \tag{37a}$$

or alternatively

$$0 = f(\tau, l, g). \tag{37b}$$

We have $n = 3$ and $d = 2$, so the problem has one dimensionless group

$$\Pi_1 = \tau l^\alpha g^\beta. \tag{38}$$

The relevant dimensions are $[\tau] = T, [l] = L, [g] = LT^{-2}$, so for Π_1 to be dimensionless equate the exponents of the dimension to find

$$\begin{aligned} 1 - 2\beta &= 0, \\ \alpha + \beta &= 0, \end{aligned}$$

which are satisfied if $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$. Thus

$$\Pi_1 = \tau\sqrt{g/l}. \tag{39}$$

We thus see Π_1 is just the constant of proportionality c from above. Thus we have

$$c = \tau\sqrt{g/l} \tag{40}$$

where c is a constant to be determined from an experiment.

3.2 Taylor's blast

This is a famous example, of some historical and fluid mechanical importance. The story goes something like this. In the early 1940's there appeared a picture of an atomic blast on the cover of Life magazine. GI Taylor, a fluid mechanician at Cambridge, wondered what the energy of the blast was. When he called his colleagues at Los Alamos and asked, they informed him that it was classified information, so he resorted to dimensional analysis. In a nuclear explosion there is an essentially instantaneous release of energy E in a small region of space. This produces a spherical shock wave, with the pressure inside the shock wave several thousands of times greater than the initial air pressure, that can be neglected. How does the radius R of this shock wave grow with time t? The relevant parameters are E, the density of air ρ and time t. Thus

$$R = R(E, \rho, t) \tag{41a}$$

or

$$0 = f(R, E, \rho, t). \tag{41b}$$

The dimensions of the physical variables are $[E] = ML^2T^{-2}, [t] = T, [R] = L$ and $[\rho] = ML^{-3}$. We have $n = 4$ physical variables and $d = 3$ dimensions, so the Pi theorem tells us there is one dimensionless group, Π_1. To form a dimensionless combination of parameters we assume

$$\Pi_1 = Et^\alpha \rho^\beta R^\gamma \tag{42}$$

and equating the exponents of dimensions in the problem requires that

$$\begin{aligned} 1+\beta &= 0, \\ \alpha - 2 &= 0, \\ 2 - 3\beta + \gamma &= 0. \end{aligned}$$

It follows that $\alpha = 2, \beta = -1$ and $\gamma = -5$, giving

$$\Pi_1 = \frac{Et^2}{\rho R^5}. \tag{43}$$

Assuming that Π_1 is constant gives

$$R = c\left(\frac{E}{\rho}\right)^{\frac{1}{5}} t^{\frac{2}{5}}. \tag{44}$$

The relation shows that if one measures the radius of the shock wave at various instants in time, the slope of the line on a log-log plot should be 2/5. The intercept of the graph would provide information about the energy E released in the explosion, if the constant c could be determined. Since information about the development of blast with time was provided by the sequence of photos on the cover of Life magazine, Taylor was able to determine the energy of the blast to be 10^{14} Joules (he estimated c to be about 1 by solving a model shock-wave problem), causing much embarrassment.

3.3 The drag on a sphere

Now what happens if you have two dimensionless groups in a problem? Let's consider the problem of the drag on a sphere. We reason that the drag on a sphere D will depend on the relative velocity, U, the sphere radius, R, the fluid density ρ and the fluid viscosity μ. Thus

$$D = D(U, R, \rho, \mu) \tag{45a}$$

or

$$0 = f(D, U, R, \rho, \mu). \tag{45b}$$

Since the physical variables are all expressible in terms of dimensions M, L and T, we have $n = 5$ and $d = 3$, so there are two dimensionless groups. There is now a certain amount of arbitrariness in determining these, however we look for combinations that make some physical sense. For our first dimensionless group, we choose the Reynolds number

$$\Pi_1 = \frac{\rho U R}{\mu}, \tag{46}$$

as we know that it arises naturally when you nondimensionalise the Navier-Stokes equations. For the second we choose the combination

$$\Pi_2 = D \rho^{\alpha} U^{\beta} R^{\gamma}, \tag{47}$$

which, if we replaced D with μ, would just give the Reynolds number. Equating the exponents of mass length and time gives, $\alpha = -1$, $\beta = -2$ and $\gamma = -2$. Thus

$$\Pi_2 = \frac{D}{\rho U^2 R^2}, \tag{48}$$

and this is called the dimensionless drag force. Buckingham's Pi theorem tells us that we must have the functional relationship

$$\Pi_2 = G(\Pi_1) \tag{49}$$

or alternatively

$$\frac{D}{\rho U^2 R^2} = G(\mathrm{Re}). \tag{50}$$

The functional dependence is determined by experiments. It is found that at high Reynolds numbers $G(\mathrm{Re}) = 1$, so that

$$D = \rho U^2 R^2. \tag{51}$$

This is known as form drag, in which resistance to motion is created by inertial forces on the sphere. At low Reynolds numbers $G(\mathrm{Re}) \propto 1/\mathrm{Re}$ so that

$$D \propto \mu U R. \tag{52}$$

This is Stokes drag, caused by the viscosity of the fluid.

The power of taking this approach can now be seen. Without dimensional analysis, to determine the functional dependence of the drag on the relevant physical variables would have required four sets of experiments to determine the functional dependence of D on velocity, radius, viscosity and density. Now we need only perform one set of experiments using our dimensionless parameters and we have all the information we need.