18   Euler equations: basic solutions and forces

In the limit where the flow is irrotational, we just need to find solutions to Laplace's equation to obtain solutions to the Euler equations. Let's write down a couple of these to gain some intuition: our aim being to acquire techniques to begin to think about airplane flight.

18.1   Point source

We know from electrostatics that a solution of Laplace's equation is just

$$\phi = -\frac{c}{4\pi r}, \tag{449}$$

where c would be the charge in an electrostatic problem. What does this solution correspond to for us? The velocity field

$$\boldsymbol{u} = \frac{c}{4\pi r^3}\boldsymbol{r} \tag{450}$$

is a radial source or sink of fluid.

18.2   Uniform flow

Another trivial solution is simply uniform flow

$$\phi = \boldsymbol{U} \cdot \boldsymbol{r}, \tag{451}$$

which works for any constant velocity vector $\boldsymbol{U}$.

18.3   Vortex solutions

We can also guess solutions by separation of variables $\phi = f(\theta)g(r)$, where θ and r denote cylindrical coordinates. Laplace's equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0. \tag{452}$$

Plugging this Ansatz into the equation gives that

$$\frac{r}{g}\frac{d}{dr}\left(r\frac{dg}{dr}\right) + \frac{1}{f}\frac{d^2 f}{d\theta^2} = 0. \tag{453}$$

Each term in (453) must be a constant, i.e.

$$\frac{d^2 f}{d\theta^2} = -fk^2, \tag{454a}$$

$$r\frac{d}{dr}\left(r\frac{dg}{dr}\right) = gk^2. \tag{454b}$$

For $k \neq 0$,

$$f = C\sin(k\theta) + D\cos(k\theta), \tag{455a}$$

with continuity of $\boldsymbol{u}$ requiring k to be an integer. Turning to the radial part we guess that $g = r^\alpha$. The radial equation then requires that $\alpha = \pm k$, giving

$$g(r) = Ar^k + Br^{-k}. \tag{455b}$$

However, if $k = 0$ then

$$f = C + D\theta. \tag{455c}$$

and the radial part is given by

$$g(r) = A + B\ln r. \tag{456}$$

So, the most general solution is

$$\phi(r,\theta) = (A_0 + B_0 \ln r)(C_0 + D_0 \theta) + \sum_{k=1}^{\infty}(C_k \sin k\theta + D_k \cos k\theta)(A_k r^k + B_k r^{-k}) \quad (457)$$

The corresponding velocity field is

$$\boldsymbol{u} = \nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}}. \quad (458)$$

Putting in our general solution with $k = 0$, we get

$$\boldsymbol{u} = \frac{B_0(C_0 + D_0\theta)}{r}\hat{\mathbf{r}} + \frac{D_0}{r}(A_0 + B_0 \ln r)\hat{\boldsymbol{\theta}} \quad (459)$$

Setting $B_0 = 0$ to obtain a continuous flow field, we get a flow with no radial component, and angular component

$$u_\theta = \frac{D}{r}, \quad (460)$$

which is irrotational by virtue of its construction. This is called a *point vortex* solution. If we consider the circulation about a loop containing the origin

$$\int \boldsymbol{u} \cdot d\mathbf{l} = \int_0^{2\pi} u_\theta r d\theta = 2\pi D = \Gamma \quad (461)$$

Thus $D = \Gamma/2\pi$, where Γ is the circulation about the point vortex.

18.4   Flow around a cylindrical wing

Okay, so this isn't the true shape of an airplane wing, but it's a good place to start. Let's see if we can calculate the lift and drag on a wing of length ℓ and radius $R \ll \ell$ moving with velocity u_0. In the frame of reference of the wing, the boundary conditions are

$$\phi \to u_0 x \quad \text{as} \quad r \to \infty, \quad (462a)$$

$$\frac{\partial\phi}{\partial r} = 0 \quad \text{at} \quad r = R. \quad (462b)$$

Using the general solution found above, the first boundary condition requires that

$$\phi = \left(u_0 r + \frac{D_1 B_1}{r}\right)\cos\theta + A_0(C_0 + D_0\theta). \quad (463)$$

The second boundary condition gives us

$$\phi = D_0\theta + u_0 \cos\theta\left(r + \frac{R^2}{r}\right), \quad (464)$$

where we have set $C_0 = 0$ since $\nabla\phi$ is all that matters. Physically we can see that $D_0 = \Gamma/2\pi$, where Γ is the circulation about the wing (check this by integrating around a circular loop containing the wing).

18.5 Forces on the circular wing

The lift and drag forces on the wing (length ℓ) are respectively given by

$$F_L = \ell \int_0^{2\pi} p(R, \theta) \, R \sin \theta d\theta, \tag{465a}$$

$$F_D = \ell \int_0^{2\pi} p(R, \theta) \, R \cos \theta d\theta. \tag{465b}$$

We can determine the pressure distribution from Bernoulli's Law

$$p = p_0 - \frac{\rho}{2}(\nabla\phi)^2\big|_{r=R} = p_0 - \frac{\rho}{2}\left(2u_0 \sin\theta - \frac{\Gamma}{2\pi R}\right)^2. \tag{466}$$

Putting this into the above relations we find that

$$F_D = 0. \tag{467}$$

This result is known as *D'Alembert's paradox*, contracting the well-known fact that drag is acting on a moving body even in in low viscosity fluids.

Furthermore, the lift on the wing is linearly proportional to the circulation about the wing;

$$F_L = \Gamma \rho \, u_0 \, \ell. \tag{468}$$

Thus, the lift on the cylindrical wing is zero (unless it is spinning, so that $\Gamma \neq 0$!). The lift force due to rotation is also known as Magnus force.

Our earlier discussion of singular perturbations suggests that *D'Alembert's paradox* for inviscid flows arises as a consequence of the fact that we have neglected viscosity in the Euler equations. Historically, this was first realized by Prandtl. Another shortcoming of our above calculations is that wings are not circular and, maybe, if we consider an alternative shape we would find lift. This will be our next avenue of investigation. We could also be worried about the fact that our problem is 2D. However, given that the aspect ratio of a wing is roughly 10:1, it is acceptable to consider 2D flow. In the next part, we will study how things change if we alter the shape of the wing. To do so will require *conformal mapping*.

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