Course 18.327 and 1.130
Wavelets and Filter Banks

Discrete-time filters: convolution; Fourier transform; lowpass and highpass filters

Discrete Time Filters

n denotes the time variable: {…, -2, -1, 0, 1, 2, …}

x[n] denotes the sequence of input values:

{…, x[-2], x[-1], x[0], x[1], x[2], …}

y[n] denotes the sequence of output values:

{…, y[-2], y[-1], y[0], y[1], y[2], …}

Assume that

a) the principle of superposition holds ⇔ system is linear, i.e. combining any two inputs in the form

$$Ax_1[n] + Bx_2[n]$$

results in an output of the form

$$Ay_1[n] + By_2[n]$$

b) the behavior of the system does not change with time, i.e. a delayed version of any input

$$x_d[n] = x[n - d]$$

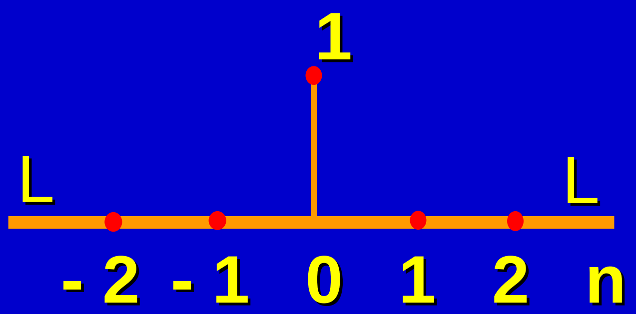
produces an output with a corresponding delay

$$y_d[n] = y[n - d]$$

Under these conditions, the system can be characterized by its response, h[n], to a unit impulse, $\delta[n]$, which is applied at time n = 0,

i.e. the particular input

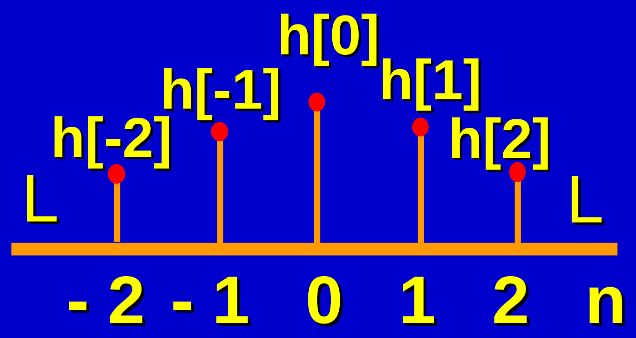
$$x[n] = \delta[n]$$



Unit Impulse

produces the output

$$y[n] = h[n]$$



Impulse Response

The general input

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

will thus produce the output

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution sum

Fourier Transform

Discrete time Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-i\omega n}$$

Inverse

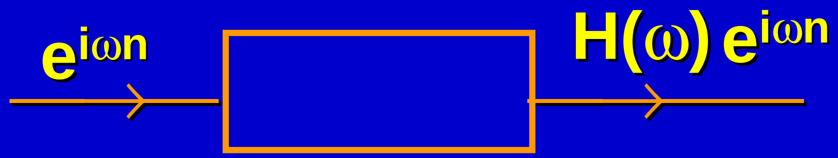
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{i\omega n}\, d\omega$$

Frequency Response

Suppose that we have the particular input

$$x[n] = e^{i\omega n}$$

What is the output?



$$y[n] = \sum_k h[k]\, x[n-k]$$

$$= e^{i\omega n} \underbrace{\sum_k h[k]\, e^{-i\omega k}}_{H(\omega)}$$

Frequency Response

Convolution Theorem

A general input

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{i\omega n}\, d\omega$$

will thus produce the output

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{X(\omega)\; H(\omega)}_{Y(\omega)}\; e^{i\omega n}\, d\omega \rightarrow \boxed{Y(\omega) = X(\omega)\, H(\omega)}$$

1 4 2 4 3

Convolution

Convolution of sequences x[n] and h[n] is denoted by

$$h[n] * x[n] = \sum_k x[k]\, h[n-k] = y[n] \text{ (say)}$$

Matrix form:

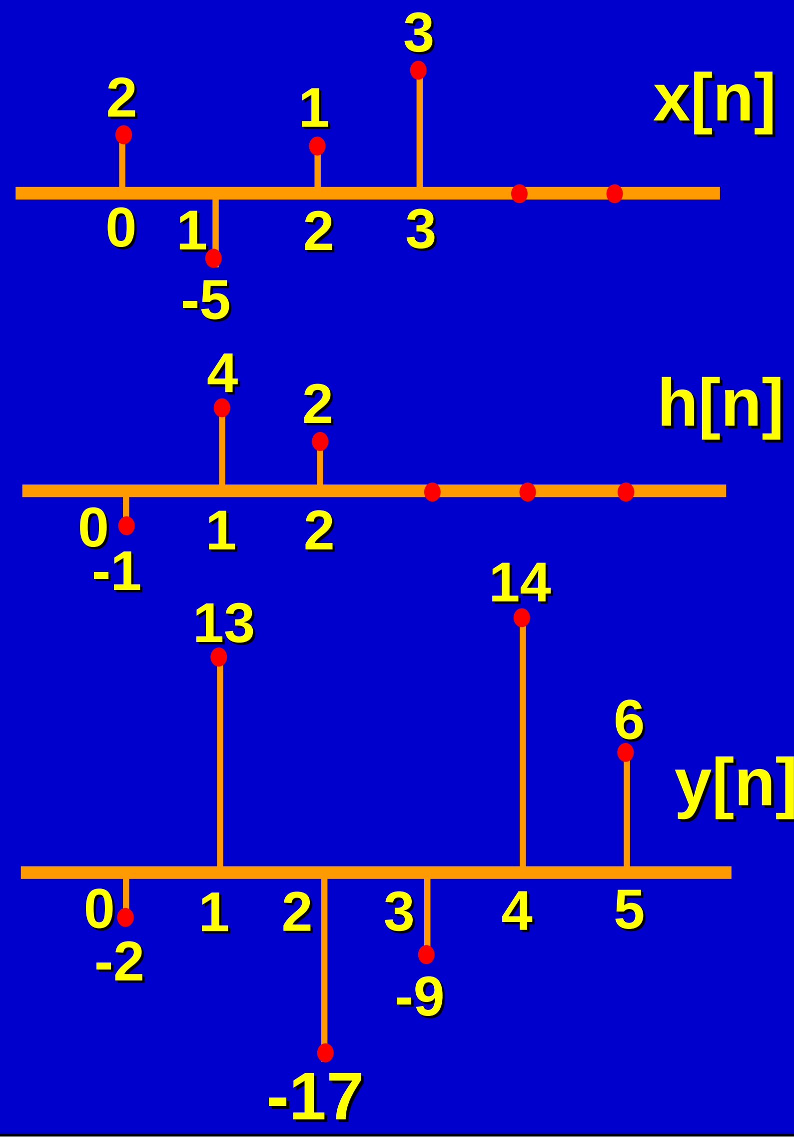
$$
\underbrace{\begin{bmatrix}
\ddots & \ddots & \ddots & & & & \\
\ddots & h[0] & h[-1] & h[-2] & & & \\
\ddots & h[1] & h[0] & h[-1] & h[-2] & & \\
 & h[2] & h[1] & h[0] & h[-1] & h[-2] & \\
 & & h[2] & h[1] & h[0] & h[-1] & \ddots \\
 & & & h[2] & h[1] & h[0] & \ddots \\
 & & & & \ddots & \ddots & \ddots
\end{bmatrix}}_{\text{Toeplitz matrix}}
\begin{bmatrix}
\vdots \\ x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
\vdots \\ y[-2] \\ y[-1] \\ y[0] \\ y[1] \\ y[2] \\ \vdots
\end{bmatrix}
$$

Convolution is the result of multiplying polynomials:

$$(\ldots + h[-1]z + h[0] + h[1]z^{-1} + \ldots)\,(\ldots + x[-1]z + x[0] + x[1]z^{-1} + \ldots) = (\ldots + y[-1]z + y[0] + y[1]z^{-1} + \ldots)$$

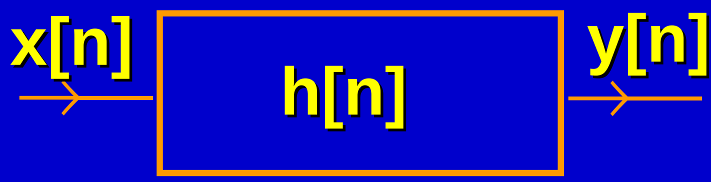
Example:

		3	1	-5	2
			2	4	-1
		-3	-1	5	-2
	12	4	-20	8	0
6	2	-10	4	0	0
6	14	-9	-17	13	-2
↑ z^{-5}	↑ z^{-4}	↑ z^{-3}	↑ z^{-2}	↑ z^{-1}	↑ z^{0}



Discrete Time Filters (summary)

Discrete Time:



$$y[n] = \sum_k x[k]\, h[n-k] \quad \text{(Convolution)}$$

Discrete –time Fourier transform

$$X(\omega) = \sum_n x[n]\, e^{-i\omega n}$$

Frequency domain representation

$$Y(\omega) = H(\omega) \bullet X(\omega) \quad \text{(Convolution theorem)}$$

Toeplitz Matrix representation:

$$
\begin{bmatrix} \vdots \\ y[-2] \\ y[-1] \\ y[0] \\ y[1] \\ y[2] \\ \vdots \end{bmatrix}
=
\begin{bmatrix}
\ddots & \ddots & \ddots & & & & \\
\ddots & h[0] & h[-1] & h[-2] & & & \\
\ddots & h[1] & h[0] & h[-1] & h[-2] & & \\
 & h[2] & h[1] & h[0] & h[-1] & h[-2] & \\
 & & h[2] & h[1] & h[0] & h[-1] & \ddots \\
 & & & h[2] & h[1] & h[0] & \ddots \\
 & & & & \ddots & \ddots & \ddots
\end{bmatrix}
\begin{bmatrix} \vdots \\ x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \\ \vdots \end{bmatrix}
$$

Filter is causal if y[n] does not depend on future values of x[n].

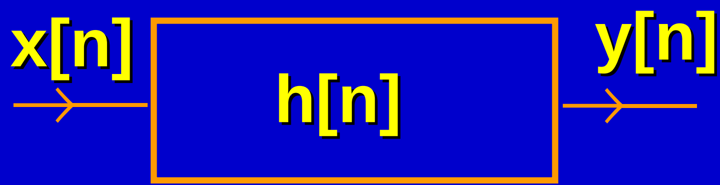
Causal filters have h[n] = 0 for n < 0.

Filters

a) **Lowpass filter example:**

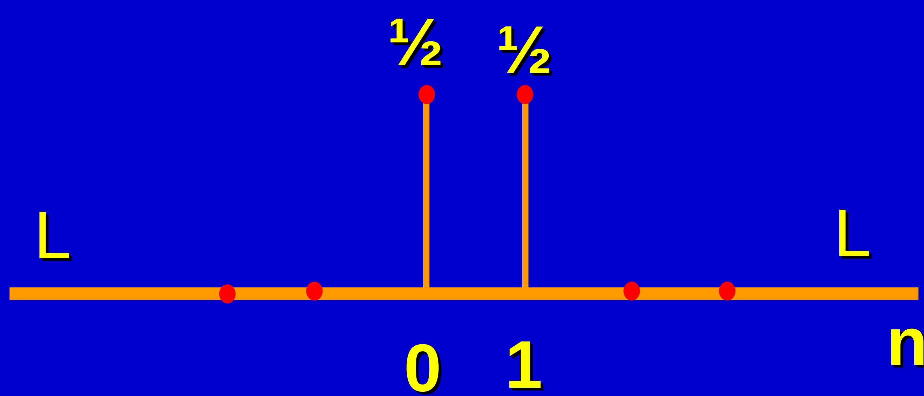
$$y[n] = \tfrac{1}{2}\, x[n] + \tfrac{1}{2}\, x[n-1]$$

Filter representation:



$$y[n] = \sum_{k} x[k]\, h[n-k]$$

Impulse response is



h[n]

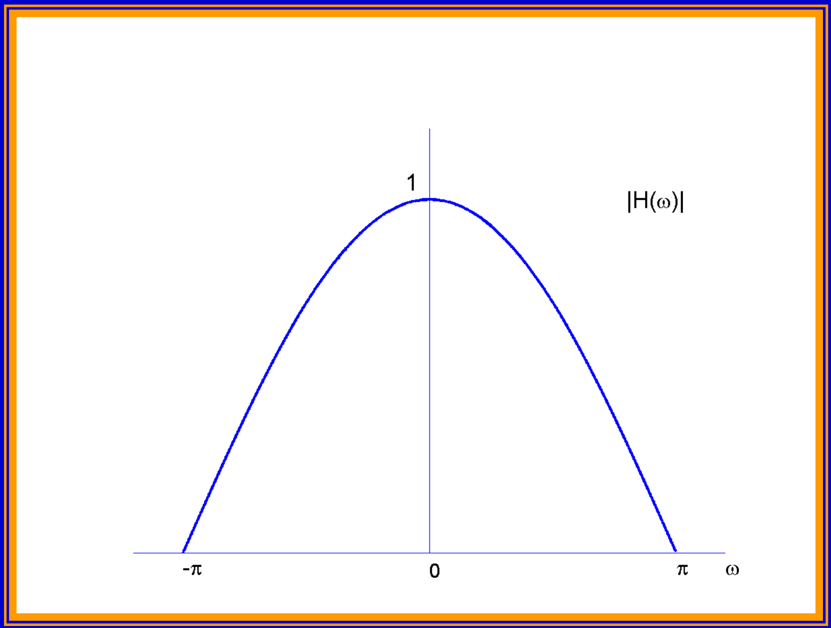
Frequency Response is

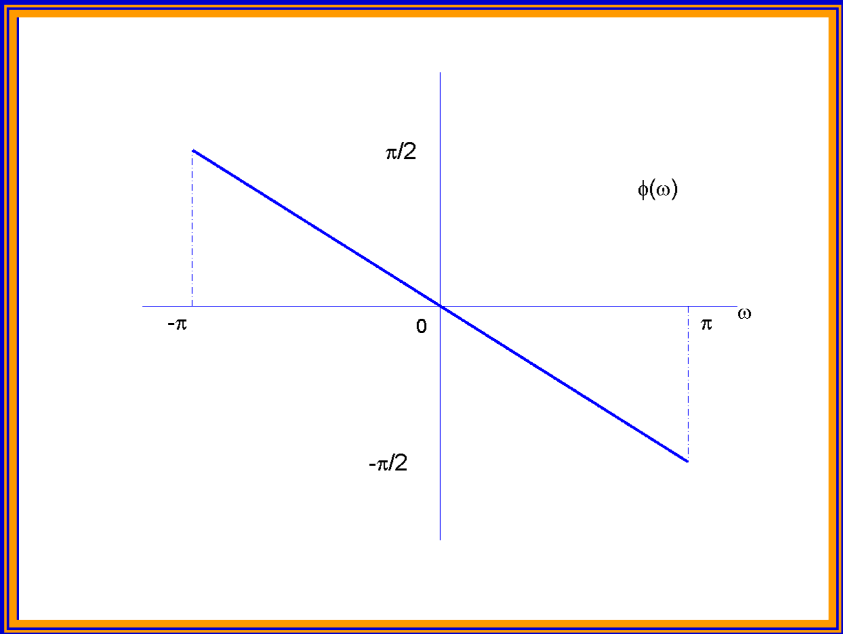
$$H(\omega) = \sum_k h[k]e^{-i\omega k}$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\, e^{-i\omega}$$

Rewrite as $H(\omega) = |H(\omega)|\, e^{i\phi(\omega)}$

$$H(\omega) = \cos(\omega/2)\, e^{-i\omega/2} \quad ; \quad -\pi \leq \omega \leq \pi$$

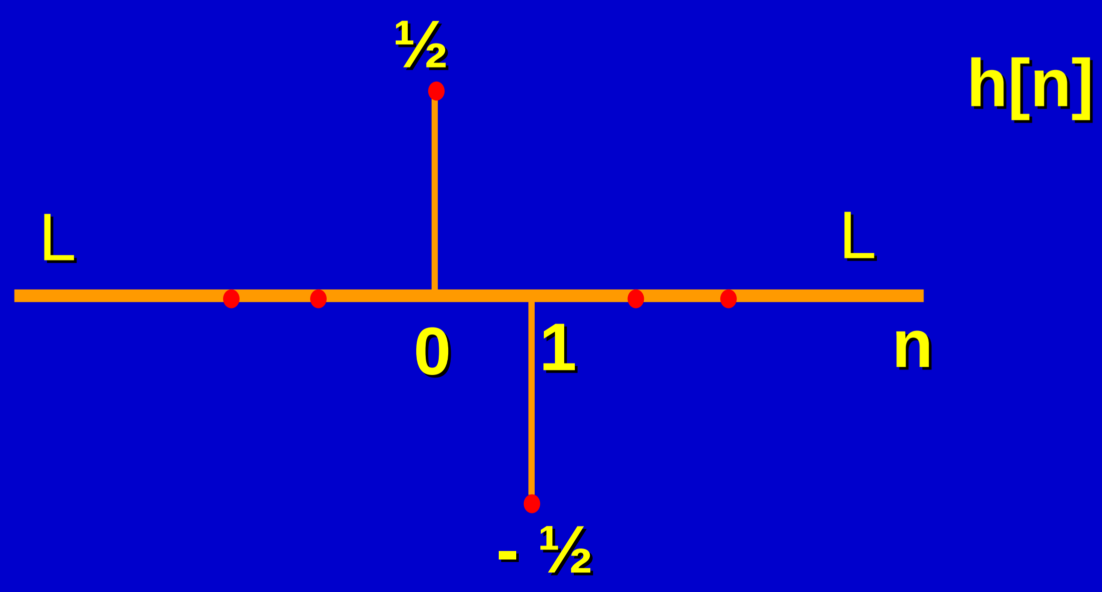




b) Highpass Filter Example

$$y[n] = \tfrac{1}{2} x[n] - \tfrac{1}{2} x[n-1]$$

Impulse response is



Frequency response is

$$H(\omega) = \tfrac{1}{2} - \tfrac{1}{2} e^{-i\omega}$$

$$= i \sin(\omega/2)\, e^{-i\omega/2}$$

$$= \begin{cases} |\sin(\omega/2)|\, e^{-i(\pi/2 + \omega/2)} & ; \ -\pi \le \omega < 0 \\ |\sin(\omega/2)|\, e^{i(\pi/2 - \omega/2)} & ; \ 0 < \omega \le \pi \end{cases}$$