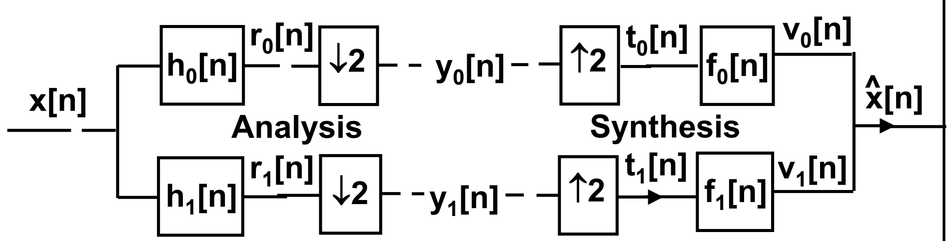
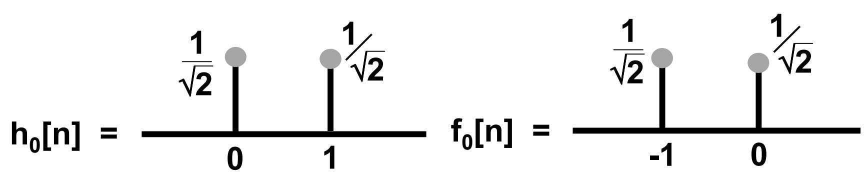
Course 18.327 and 1.130
Wavelets and Filter Banks

Filter Banks: time domain (Haar example) and frequency domain; conditions for alias cancellation and no distortion

Haar Filter Bank

Simplest (non-trivial) example of a two channel FIR perfect reconstruction filter bank.





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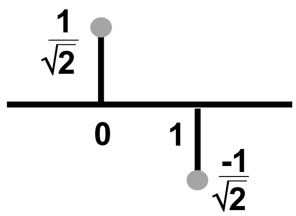
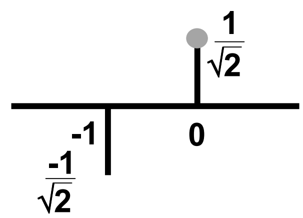
$h_1[n]$ =



$f_1[n]$ =



Analysis:

$r_0[n] = \frac{1}{\sqrt{2}}\,(x[n] + x[n-1])$ **lowpass filter**

$y_0[n] = r_0[2n]$ **downsampler**

$y_0[n] = \frac{1}{\sqrt{2}}\,(x[2n] + x[2n-1])$ -----------------①

Similarly

$y_1[n] = \frac{1}{\sqrt{2}}\,(x[2n] - x[2n-1])$ ------------------②

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Matrix form

$$\begin{bmatrix} \vdots \\ y_0[0] \\ y_0[1] \\ \vdots \\ \hline \vdots \\ y_1[0] \\ y_1[1] \\ \vdots \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} & & \vdots & & & \\ \cdots & 1 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 1 & \cdots \\ \hline & & \vdots & & & \\ \cdots & -1 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & -1 & 1 & \cdots \\ & & \vdots & & & \end{bmatrix} \begin{bmatrix} \vdots \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \\ \vdots \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} y_o \\ \hline y_1 \end{bmatrix} = \begin{bmatrix} L \\ \hline B \end{bmatrix} x$$ -------------------③

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Synthesis

$$t_0[n] = \begin{cases} y_0[n/2] & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad \text{upsampler}$$

$$v_0[n] = \tfrac{1}{\sqrt{2}} \left(t_0[n+1] + t_0[n] \right) \qquad \text{lowpass filter}$$

$$= \begin{cases} \frac{1}{\sqrt{2}} y_0[n/2] & n \text{ even} \\ \frac{1}{\sqrt{2}} y_0[\frac{n+1}{2}] & n \text{ odd} \end{cases}$$

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Similarly

$$v_1[n] = \begin{cases} \frac{1}{\sqrt{2}} y_1[n/2] & n \text{ even} \\ -\frac{1}{\sqrt{2}} y_1[\frac{n+1}{2}] & n \text{ odd} \end{cases}$$

So, the reconstructed signal is

$$\hat{x}[n] = v_0[n] + v_1[n]$$

$$= \begin{cases} \tfrac{1}{\sqrt{2}} \left(y_0[n/2] + y_1[n/2] \right) & n \text{ even} \\ \tfrac{1}{\sqrt{2}} \left(y_0[\frac{n+1}{2}] - y_1[\frac{n+1}{2}] \right) & n \text{ odd} \end{cases}$$

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i.e.

$$\hat{x}[2n-1] = \tfrac{1}{\sqrt{2}}\,(y_0[n] - y_1[n]) = x[2n-1]$$

from ① and ②

$$\hat{x}[2n] = \frac{1}{\sqrt{2}}\,(y_0[n] + y_1[n]) = x[2n]$$

So $\hat{x}[n] = x[n] \Rightarrow$ **Perfect reconstruction!**

In general, we will make all filters causal, so we will have

$$\hat{x}[n] = x[n - n_0] \quad \Rightarrow \text{ PR with delay}$$

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Matrix form

$$\begin{bmatrix} \vdots \\ \hat{x}[-1] \\ \hat{x}[0] \\ \\ \hat{x}[1] \\ \hat{x}[2] \\ \vdots \end{bmatrix} = \frac{1}{\sqrt{2}} \left[\begin{array}{cccc|cccc} & \vdots & \vdots & & & \vdots & \vdots & \\ & 1 & 0 & & & -1 & 0 & \\ & 1 & 0 & & & 1 & 1 & \\ \cdots & & & \cdots & \cdots & & & \cdots \\ & 0 & 1 & & & 0 & -1 & \\ & 0 & 1 & & & -1 & 1 & \\ & \vdots & \vdots & & & \vdots & \vdots & \end{array}\right] \left[\begin{array}{c} \vdots \\ y_0[0] \\ y_0[1] \\ \vdots \\ \hline \vdots \\ y_1[0] \\ y_1[1] \\ \vdots \end{array}\right]$$

$$\hat{x} = \left[\begin{array}{c|c} L^T & B^T \end{array}\right] \left[\begin{array}{c} y_0 \\ \hline y_1 \end{array}\right] \qquad \text{----------------④}$$

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Perfect reconstruction means that the synthesis bank is the inverse of the analysis bank.

$$\hat{x} = x \Rightarrow \underbrace{\left[\begin{array}{c|c} L^T & B^T \end{array}\right]}_{W^{-1}} \underbrace{\left[\begin{array}{c} L \\ \hline B \end{array}\right]}_{W} = I$$

(Wavelet transform matrix)

In the Haar example, we have the special case

$$W^{-1} = W^T \rightarrow \text{orthogonal matrix}$$

So we have an orthogonal filter bank, where
Synthesis bank = Transpose of Analysis bank

$f_0[n] \quad = \quad h_0[-n]$

$f_1[n] \quad = \quad h_1[-n]$

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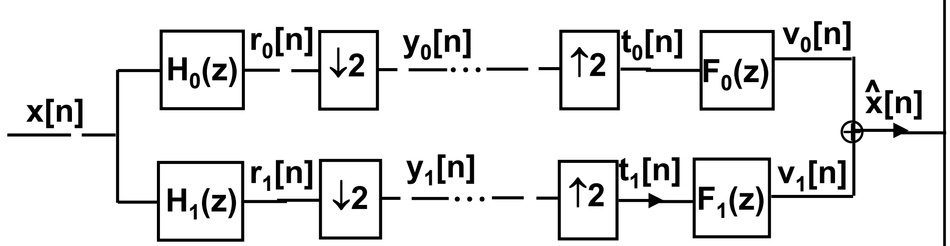
Perfect Reconstruction Filter Banks

General two-channel filter bank



z-transform definition:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Put $z = e^{i\omega}$ to get DTFT

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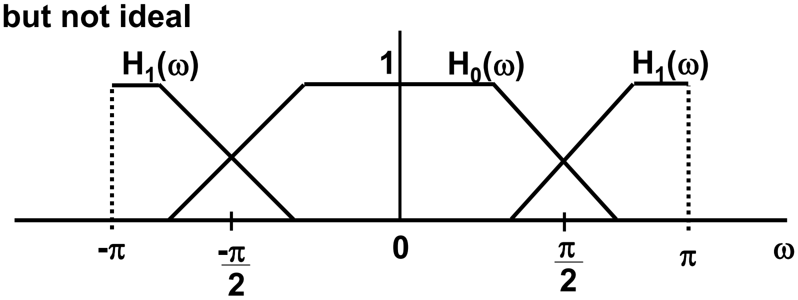
Perfect reconstruction requirement:

$\hat{x}[n] = x[n - \ell]$ (ℓ time delays)

$\hat{X}(z) = z^{-\ell} X(z)$

$H_0(z)$ and $H_1(z)$ are normally lowpass and highpass, but not ideal



⇒ Downsampling operation in each channel can produce aliasing

Let's see why:

Lowpass channel has

$$Y_0(z) = \tfrac{1}{2}\{R_0(z^{1/2}) + R_0(-z^{1/2})\} \quad \text{(downsampling)}$$

$$= \tfrac{1}{2}\{H_0(z^{1/2})\, X(z^{1/2}) + H_0(-z^{1/2})X(-z^{1/2})\}$$

In frequency domain:

$X(z) \rightarrow X(\omega)$ or $X(e^{i\omega})$

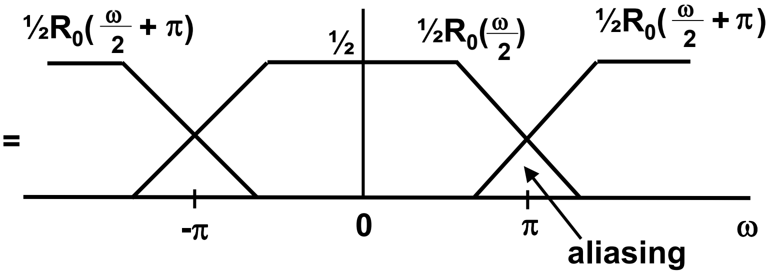
$X(-z) \rightarrow X(\omega + \pi)$

$X(z^{1/2}) \rightarrow X(\frac{\omega}{2})$

$$Y_0(\omega) = \tfrac{1}{2}\{H_0(\tfrac{\omega}{2})\, X(\tfrac{\omega}{2}) + H_0(\tfrac{\omega}{2} + \pi)X(\tfrac{\omega}{2} + \pi)\}$$

Suppose $X(\omega) = 1$ (input has all frequencies)
Then $R_0(\omega) = H_0(\omega)$, so that after downsampling we have

$Y_0(\omega) =$



Goal is to design $F_0(z)$ and $F_1(z)$ so that the overall system is just a simple delay - with no aliasing term:

$$V_0(z) + V_1(z) = z^{-\ell} X(z)$$

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$$V_0(z) = F_0(z)\, T_0(z)$$

$$= F_0(z)\, Y_0(z^2) \qquad \text{(upsampling)}$$

$$= \tfrac{1}{2} F_0(z)\{ H_0(z)\, X(z) + H_0(-z)\, X(-z)\}$$

$$V_1(z) = \tfrac{1}{2} F_1(z)\{ H_1(z)\, X(z) + H_1(-z)\, X(-z)\}$$

So we want

$$\tfrac{1}{2}\{F_0(z)\, H_0(z) + F_1(z)\, H_1(z)\}\, X(z)$$
$$+ \qquad = z^{-\ell} X(z)$$
$$\tfrac{1}{2}\{F_0(z)\, H_0(-z) + F_1(z)\, H_1(-z)\}\, X(-z)$$

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Compare terms in X(z) and X(-z):

1) **Condition for no distortion (terms in X (z) amount to a delay)**

$$F_0(z)\,H_0(z) + F_1(z)\,H_1(z) = 2z^{-\ell} \quad \text{--------------①}$$

2) **Condition for alias cancellation (no term in X(-z))**

$$F_0(z)\,H_0(-z) + F_1(z)\,H_1(-z) = 0 \quad \text{--------------②}$$

To satisfy alias cancellation condition, choose

$$\begin{aligned} F_0(z) &= H_1(-z) \\ F_1(z) &= -H_0(-z) \end{aligned} \quad \text{----------------------③}$$

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What happens in the time domain?

$$F_0(z) = H_1(-z) \qquad\qquad F_0(\omega) = H_1(\omega + \pi)$$

$$= \sum_n h_1[n]\,(-z)^{-n}$$

$$= \sum_n (-1)^n\,h_1[n]\,z^{-n}$$

So the filter coefficients are

$f_0[n] = (-1)^n \; h_1[n]$ **alternating signs**

$f_1[n] = (-1)^{n+1}\,h_0[n]$ **rule**

Example

$h_0[n] = \{ a_0, a_1, a_2\}$ ↘ $f_1[n] = \{-a_0, a_1, -a_2\}$

$h_1[n] = \{ b_0, b_1, b_2\}$ ↗ $f_0[n] = \{ b_0, -b_1, b_2 \}$

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Product Filter

Define

$$P_0(z) = F_0(z)\,H_0(z) \quad \text{------------------------------④}$$

Substitute $F_1(z) = -H_0(-z)$, $H_1(z) = F_0(-z)$

in the zero distortion condition (Equation ①)

$$F_0(z)\,H_0(z) - F_0(-z)\,H_0(-z) = 2z^{-\ell}$$

i.e. $$P_0(z) - P_0(-z) = 2z^{-\ell} \quad \text{----------------------------⑤}$$

Note: ℓ must be odd since LHS is an odd function.

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Normalized Product Filter

Define

$$P(z) = z^{\ell}\,P_0(z) \quad \text{----------------------------⑥}$$

$$P(-z) = -z^{\ell}\,P_0(-z) \quad \text{since } \ell \text{ is odd}$$

So we can rewrite Equation ⑤ as

$$z^{-\ell}\,P(z) + z^{-\ell}\,P(-z) = 2z^{-\ell}$$

i.e. $$P(z) + P(-z) = 2 \quad \text{---------------------------⑦}$$

This is the condition on the normalized product filter for Perfect Reconstruction.

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Design Process

1. Design $P(z)$ to satisfy Equation ⑦. This gives $P_0(z)$. Note: $P(z)$ is designed to be lowpass.
2. Factor $P_0(z)$ into $F_0(z)\,H_0(z)$. Use Equations ③ to find $H_1(z)$ and $F_1(z)$.

Note: Equation ⑦ requires all even powers of z (except z^0) to be zero:

$$\sum_n p[n]z^{-n} + \sum_n p[n](-z)^{-n} = 2$$

$$\Rightarrow \; p[n] = \begin{cases} 1 & ; \quad n = 0 \\ 0 & ; \quad \text{all even } n \ (n \neq 0) \end{cases}$$

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For odd n, p[n] and –p[n] cancel.

The odd coefficients, p[n], are free to be designed according to additional criteria.

Example: Haar filter bank

$$H_0(z) = \frac{1}{\sqrt{2}}(1 + z^{-1}) \qquad H_1(z) = \frac{1}{\sqrt{2}}(1 - z^{-1})$$

$$F_0(z) = H_1(-z) = \frac{1}{\sqrt{2}}(1 + z^{-1})$$

$$F_1(z) = -H_0(-z) = \frac{1}{\sqrt{2}}(1 - z^{-1})$$

$$P_0(z) = F_0(z)\,H_0(z) = \frac{1}{2}(1 + z^{-1})^2$$

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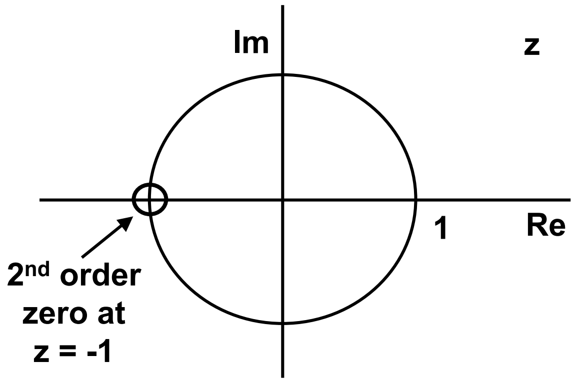
So the Perfect Reconstruction requirement is

$$P_0(z) – P_0(-z) = \frac{1}{2}\, 1 + 2z^{-1} + z^{-2}) - \frac{1}{2}\, 1 – 2z^{-1} + z^{-2})$$

$$= 2z^{-1} \qquad \Rightarrow \qquad \ell = 1$$

$$P(z) = z^{\ell}\, P_0(z) = \frac{1}{2}(1 + z)(1 + z^{-1})$$



Zeros of P(z):
$1 + z = 0$
$1 + z^{-1} = 0$

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