

Course 18.327 and 1.130 Wavelets and Filter Banks

Accuracy of wavelet approximations
(Condition A); vanishing moments;
polynomial cancellation in filter banks

Accuracy of Wavelet Approximations

Biorthogonal case:

$$\tilde{\phi}(t) = 2 \sum_n h_0[-n] \tilde{\phi}(2t - n) \quad \phi(t) = 2 \sum_n f_0[n] \phi(2t - n)$$

$$\tilde{w}(t) = 2 \sum_n h_1[-n] \tilde{\phi}(2t - n) \quad w(t) = 2 \sum_n f_1[n] \phi(2t - n)$$

Biorthogonality means

$$\int_{-\infty}^{\infty} \phi(t) \tilde{\phi}(t - n) dt = \delta[n] \quad \int_{-\infty}^{\infty} w(t) \tilde{w}(t - n) dt = \delta[n]$$

$$\int_{-\infty}^{\infty} \phi(t) \tilde{w}(t - n) dt = 0 \quad \int_{-\infty}^{\infty} w(t) \tilde{\phi}(t - n) dt = 0$$

Suppose that $F_0(z)$ has p zeros at $z = -1$

$$H_1(z) = F_0(-z) \quad \rightarrow \quad p \text{ zeros at } z = 1$$

i.e.

$$\frac{\partial^\ell}{\partial z^\ell} H_1(z) \Big|_{z=1} = 0 \quad \text{for } \ell = 0, 1, 2, \dots, p-1$$

$$H_1(z) \Big|_{z=1} = \sum_n h_1[n] z^{-n} \Big|_{z=1} = \sum_n h_1[n]$$

$$\frac{\partial}{\partial z} H_1(z) \Big|_{z=1} = \sum_n (-n) h_1[n] z^{-n-1} \Big|_{z=1} = -\sum_n n h_1[n]$$

$$\frac{\partial^2}{\partial z^2} H_1(z) \Big|_{z=1} = \sum_n (-n)(-n-1) h_1[n] z^{-n-2} \Big|_{z=1} = \sum_n n^2 h_1[n] + \sum_n n h_1[n]$$

etc.

So

$$\sum_n n^\ell h_1[n] = 0 \quad \text{for } \ell = 0, 1, 2, \dots, p-1$$

3

Consider the moments of the analysis wavelet:

$$\begin{aligned} \int_{-\infty}^{\infty} t^\ell \tilde{w}(t) dt &= 2 \sum_n h_1[-n] \int_{-\infty}^{\infty} t^\ell \tilde{\phi}(2t - n) dt \\ &= 2 \sum_n h_1[-n] \int_{-\infty}^{\infty} \left(\frac{\tau+n}{2}\right)^\ell \tilde{\phi}(\tau) d\tau / 2 \\ &= \frac{1}{2^\ell} \sum_n h_1[-n] \int_{-\infty}^{\infty} \sum_{i=0}^{\ell} \binom{\ell}{i} \tau^{\ell-i} n^i \tilde{\phi}(\tau) d\tau \\ &= \frac{1}{2^\ell} \sum_{i=0}^{\ell} \binom{\ell}{i} \underbrace{\left(\sum_n h_1[-n] n^i \right)}_0 (-1)^i \int_{-\infty}^{\infty} \tau^{\ell-i} \tilde{\phi}(\tau) d\tau \\ &\quad \text{if } 0 \leq i < p \end{aligned}$$

So the analysis wavelet has p vanishing moments:

$$\int_{-\infty}^{\infty} t^\ell \tilde{w}(t) dt = 0 \quad \text{for } \ell = 0, 1, 2, \dots, p-1$$

4

What do vanishing moments mean?

Try expanding the polynomial $\sum_{\ell=0}^{p-1} \alpha_{\ell} t^{\ell}$ in a wavelet basis:

$$P(t) \equiv \sum_{\ell=0}^{p-1} \alpha_{\ell} t^{\ell} = \sum_k c_{0,k} \phi_{0,k}(t) + \sum_{j \geq 0} \sum_k d_{j,k} w_{j,k}(t)$$

Then

$$d_{j,k} = \int_{-\infty}^{\infty} P(t) \tilde{w}_{j,k}(t) dt = \sum_{\ell=0}^{p-1} \alpha_{\ell} 2^{j\ell/2} \int_{-\infty}^{\infty} t^{\ell} \tilde{w}(2^j t - k) dt = 0$$

i.e. polynomials of degree $p - 1$ can be expressed as a linear combination of scaling functions:

$$\sum_{\ell=0}^{p-1} \alpha_{\ell} t^{\ell} = \sum_k c_{0,k} \phi(t - k) \text{ for some } c_{0,k}$$

5

Example (orthogonal wavelets)

$$f(t) = t^{\ell}$$

$$f_0(t) = \sum_k a_0[k] \phi(t - k) \in V_0 ; a_0[k] = \int_{-\infty}^{\infty} t^{\ell} \phi(t - k) dt$$

$$g_0(t) = \sum_k b_0[k] w(t - k) \in W_0 ; b_0[k] = \int_{-\infty}^{\infty} t^{\ell} w(t - k) dt$$

Suppose that $\phi(t)$ comes from a spline of degree $p - 1$

($h_0[n]$ has p zeros at π) with $p - 1 \geq \ell$.

Then we can write

$$t^{\ell} = \sum_k a[k] \phi(t - k)$$

The expansion coefficients are easily found since

$\phi(t - k)$ are orthonormal:

$$a[k] = \int_{-\infty}^{\infty} t^{\ell} \phi(t - k) dt$$

6

Also, since $V_j \perp W_j$, we have

$$\int_{-\infty}^{\infty} t^\ell w(t-n) dt = \sum_k a[k] \int_{-\infty}^{\infty} \phi(t-k) w(t-n) dt = 0$$

→ vanishing moment property

So we have

$$f_0(t) = t^\ell$$

$$g_0(t) = 0$$

$$V_{j+1} = V_j \oplus W_j \Rightarrow f_{j+1}(t) = f_j(t) + g_j(t)$$

$$f_1(t) = f_0(t) + g_0(t) = t^\ell$$

All $f_j(t)$ are the same as $f(t)$!

7

Polynomial Data

Suppose $x[n] = 1$ for $n \geq 0$ (unit step).

$$X(z) = \sum_{n=0}^{\infty} z^{-n}$$

$$\frac{d}{dz} X(z) = -\sum_{n=0}^{\infty} n z^{-n-1}$$

$$\frac{d^2}{dz^2} X(z) = \sum_{n=0}^{\infty} n(n+1) z^{-n-2}$$

⋮

$$\frac{d^k}{dz^k} X(z) = (-1)^k \sum_{n=0}^{\infty} n(n+1)\dots(n+k-1) z^{-n-k}$$

8

But we know that

$$X(z) = \frac{1}{1 - z^{-1}} \quad ; \quad |z| > 1$$

So

$$\frac{d}{dz} X(z) = \frac{-z^{-2}}{(1 - z^{-1})^2}$$

$$\frac{d^2}{dz^2} X(z) = \frac{2z^{-3}}{(1 - z^{-1})^3}$$

$$\vdots \quad \quad \quad \vdots$$

$$\frac{d^k}{dz^k} X(z) = \frac{(-1)^k k! z^{-k-1}}{(1 - z^{-1})^{k+1}}$$

9

So if

$$x[n] = n(n + 1)(n + 2) \dots (n + p - 2) \quad ; \quad n \geq 0$$

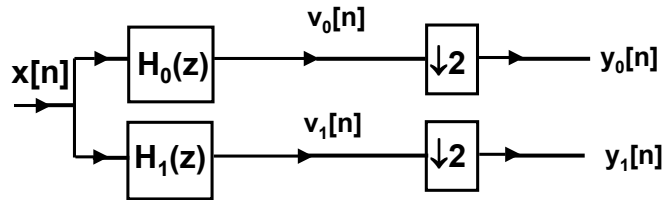
polynomial of degree $p - 1$

then

$$X(z) = \frac{(p - 1)! z^{-1}}{(1 - z^{-1})^p} \quad ; \quad |z| > 1$$

10

Polynomial Data and Condition A_p



Consider Daubechies' filters

$$\begin{aligned} H_0(z) &= (1 + z^{-1})^p Q(z) && p \text{ zeros at } z = -1 \\ H_1(z) &= -z^{-N} H_0(-z^{-1}) \\ &= (1 - z^{-1})^p R(z) && ; R(z) = (-1)^{p-1} z^{-N+p} Q(-z^{-1}) \end{aligned}$$

Suppose that the input data is a polynomial of degree $p-1$:

$$x[n] = \sum_{k=0}^{p-1} a[k] S_p[n - k] \quad \text{combination of shifts of } S_p[n]$$

where

$$S_p[n] = n(n+1)(n+2)\dots(n+p-2) \quad \text{for } n \geq 0$$

11

z -transform is

$$\begin{aligned} X(z) &= \sum_{k=0}^{p-1} a[k] \frac{z^{-k}(p-1)! z^{-1}}{(1-z^{-1})^p} \\ &= \frac{(p-1)! z^{-1} A(z)}{(1-z^{-1})^p} && ; |z| > 1 \end{aligned}$$

Lowpass channel:

$$V_0(z) = H_0(z) X(z) = \frac{(p-1)! z^{-1} A(z)(1+z^{-1})^p Q(z)}{(1-z^{-1})^p}$$

So

$v_0[n]$ is a polynomial of degree $p - 1$.

$$y_0[n] = v_0[2n]$$

\Rightarrow $y_0[n]$ is a polynomial of degree $p - 1$.

12

Highpass channel:

$$\begin{aligned} V_1(z) = H_1(z) X(z) &= \frac{(p-1)! z^{-1} A(z)(1-z^{-1})^p R(z)}{(1-z^{-1})^p} \\ &= (p-1)! z^{-1} A(z)R(z) \end{aligned}$$

So $v_1[n]$ has finite length (even though $x[n]$ has infinite length.)

$$y_1[n] = v_1[2n]$$

$\Rightarrow y_1[n]$ has finite length

i.e.

$y_1[n] = 0$, except for startup/boundary effects.