1. Let H be a Hilbert space, and let $\{e_n\}_{n=1}^{\infty}$ be a countably infinite orthonormal subset of H. Let $W = \text{span}\{e_n\}_{n=1}^{\infty}$ be the subspace of all **finite** linear combinations of these vectors.

   (a) Prove that $w \in \overline{W}$ (the closure of W) if and only if there exists $\{c_n\}_{n=1}^{\infty} \in \ell^2$ such that

   $$w = \sum_{n=1}^{\infty} c_n e_n.$$

   (b) Let $u \in H$. Prove that for all $w \in \overline{W}$,

   $$u - \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n \quad \leq \|u - w\|$$

   with equality if and only if $w = \sum_{n=1}^{\infty} \langle u, e_n \rangle e_n$.
   Hint: If $w = \sum_{n=1}^{\infty} c_n e_n$, then compute $\|u - w\|^2$ and complete the square.

2. Let H be a Hilbert space, and let W be a linear subspace of H.

   (a) Prove that

   $$W^{\perp} := \{u \in H \mid \langle u, w \rangle = 0 \text{ for all } w \in W\}$$

   is a closed linear subspace of H.

   (b) Prove that $\overline{W} = (W^{\perp})^{\perp}$.

3. Let $s \geq 0$. We say $f \in L^2([-\pi, \pi])$ is an element of the *Sobolev space* $H^s(\mathbb{T})$ of order s if

   $$\lim_{N \to \infty} \sum_{|n| \leq N} |\hat{f}(n)|^2 (1 + |n|^2)^s < \infty.$$

   (a) Suppose that $f \in C^k([-\pi, \pi])$ and

   $$f(\pi) = f(-\pi), f'(\pi) = f'(-\pi), \ldots, f^{(k-1)}(\pi) = f^{(k-1)}(-\pi).$$

   Prove that $f \in H^s(\mathbb{T})$ for all $0 \leq s \leq k$.
   Hint: Integrate by ...

   (b) Prove that $H^s(\mathbb{T})$ is a vector space and

   $$\langle f, g \rangle_{H^s(\mathbb{T})} := \lim_{N \to \infty} \sum_{|n| \leq N} \hat{f}(n) \overline{\hat{g}(n)} (1 + |n|^2)^s, \quad f, g \in H^s,$$

   is a Hermitian inner product on $H^s(\mathbb{T})$.

   (c) Prove that $H^s(\mathbb{T})$ is a Hilbert space.

4. (Sobolev embedding) Let $s > 1/2$. The purpose of this exercise is to show that $H^s(\mathbb{T})$ *embeds* into $C([-\pi, \pi])$ continuously.

   (a) Prove that there exists a constant $C(s) > 0$ such that for all $f \in H^s(\mathbb{T})$

   $$\lim_{N\to\infty} \sum_{|n|\leq N} |\hat{f}(n)| \leq C(s)\|f\|_{H^s(\mathbb{T})}.$$

   (b) Prove that if $f \in H^s(\mathbb{T})$ then $f \in C([-\pi, \pi])$ (strictly speaking, $\exists g \in C([-\pi, \pi])$ such that $f = g$ a.e.), and moreover,

   $$\|f\|_\infty \leq C(s)\|f\|_{H^s(\mathbb{T})}.$$

   Hint: It may be useful to recall the Weierstrass M-test from 18.100.