Assignments are to be submitted to Gradescope by 24:00.

In what follows, you may use without proof the following theorem (you proved the case $p = 1$ in the previous assignment).

Theorem. *Let $a < b$, $1 \leq p < \infty$ and $f \in L^p([a,b])$. For all $\epsilon > 0$ there exist a step function $\psi \in L^p([a,b])$ and $g \in C([a,b])$ with $g(a) = g(b) = 0$ such that*

$$\|f - \psi\|_p + \|f - g\|_p < \epsilon.$$

1. Let $1 \leq p < \infty$.
   (a) Prove that for all $1 \leq q \leq p$, $L^p([a,b]) \subset L^q([a,b])$.
   (b) Prove that $L^p([a,b])$ is separable.
   Hint: See the second lecture of the week for an outline of the proof.
   (c) Let $f \in L^p(\mathbb{R})$, and let $\epsilon > 0$. Prove that there exists $g \in C(\mathbb{R})$ with the property that there exists $R > 0$ such that for all $|x| > R$, $g(x) = 0$, and
   $$\|f - g\|_p < \epsilon.$$
   Hint: Prove that $\|f\chi_{[-n,n]} - f\|_p \to 0$ as $n \to \infty$ and use the theorem above.
   (d) Prove that $L^p(\mathbb{R})$ is separable.

2. (a) Let $E \subset \mathbb{R}$ be measurable. Prove that $L^\infty(E)$ is a Banach space.
   (b) Let $a < b$. Prove that if $f \in C([a,b])$ then $\|f\|_{L^\infty([a,b]} = \|f\|_\infty$, i.e.
   $$\inf\{C \geq 0 \mid m(\{x \in [a,b] \mid |f(x)| > C\}) = 0\} = \sup_{x \in [a,b]} |f(x)|.$$
   Conclude that $C([a,b])$ is not dense in $L^\infty([a,b])$.

3. Let $a < b$, $1 \leq p \leq \infty$ and $g \in L^\infty([a,b])$. Define
   $$Tf(x) := f(x)g(x), \quad f \in L^p([a,b]).$$
   Prove that $T \in \mathcal{B}(L^p([a,b]), L^p([a,b]))$ and $\|T\| = \|g\|_{L^\infty}$.

4. (a) Let H be a pre-Hilbert space. Prove the polarization identity: for all $u, v \in H$
   $$\langle u, v \rangle = \frac{1}{4}\Big[\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2\Big]. \qquad (\dagger)$$
   (b) Suppose that H is a normed space with norm $\|\cdot\|$ satisfying the parallelogram law. Prove that there exists a Hermitian inner product $\langle \cdot, \cdot \rangle$ on H such that for all $u \in H$, $\|u\| = \langle u, u \rangle^{1/2}$.
   Hint: Define $\langle u, v \rangle$ via the right-hand side of $(\dagger)$ and prove that $\langle \cdot, \cdot \rangle$ is a Hermitian inner product.

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