Assignments are to be submitted to Gradescope. In what follows, we denote the Lebesgue integral of an integrable function f over $[a,b]$ via

$$\int_a^b f(x)dx.$$

1. Let $a < b$. A simple function $\psi : [a,b] \to \mathbb{R}$ with range $\psi([a,b]) = \{a_1, \dots, a_n\}$ is a *step function* if for all $i = 1, \dots, n$, $\psi^{-1}(\{a_i\})$ is a finite union of intervals. This is equivalent to

$$\psi = \sum_{i=1}^n a_i \chi_{U_i},$$

where for all i, $U_i \subset [a,b]$ is a finite union of intervals, for all $i \neq j$, $U_i \cap U_j = \varnothing$ and $\cup_{i=1}^n U_i = [a,b]$.

(a) Suppose that $\psi : [a,b] \to \mathbb{R}$ is a step function such that

$$\sup_{x\in[a,b]} |\psi(x)| \le B$$

Let $\epsilon > 0$. Prove that there exists a continuous function $g : [a,b] \to \mathbb{R}$ and a measurable set $E \subset [a,b]$ such that,

$$g(a) = g(b) = 0, \quad \sup_{x\in[a,b]} |g(x)| \le B, \quad m(E) < \epsilon,$$
$$\forall x \in E^c, \quad |\psi(x) - g(x)| < \epsilon.$$

(b) Suppose that $\varphi : [a,b] \to \mathbb{R}$ is a simple function such that

$$\sup_{x\in[a,b]} |\varphi(x)| \le B.$$

Let $\epsilon > 0$. Prove that there exists a step function $\psi : [a,b] \to \mathbb{R}$ and a measurable set $E \subset [a,b]$ such that

$$\sup_{x\in[a,b]} |\psi(x)| \le B, \quad m(E) < \epsilon,$$
$$\forall x \in E^c, \quad |\varphi(x) - \psi(x)| < \epsilon.$$

Hint: See Littlewood's first principle from Assignment 4.

2. Let $a < b$. Suppose that $f : [a,b] \to \mathbb{R}$ is a measurable function such that

$$\sup_{x\in[a,b]} |f(x)| \le B.$$

Let $\epsilon > 0$. Prove that there exists a continuous function $g : [a,b] \to \mathbb{R}$ and a measurable set $E \subset [a,b]$ such that,

$$g(a) = g(b) = 0, \quad \sup_{x\in[a,b]} |g(x)| \le B, \quad m(E) < \epsilon,$$
$$\forall x \in E^c, \quad |f(x) - g(x)| < \epsilon.$$

This result is known as *Littlewood's third principle*: every measurable function is nearly continuous.

3. Let $a < b$. Suppose that $f : [a, b] \to \mathbb{R}$ is Lebesgue integrable.

   (a) Let $\epsilon > 0$. Prove that there exists a bounded measurable function $h : [a, b] \to \mathbb{R}$ such that

   $$\int_a^b |f(x) - h(x)|dx < \epsilon.$$

   Hint: Let $E_n = f^{-1}([-n, n])$ and $h_n = f\chi_{E_n}$. Prove that $\lim_{n\to\infty} \int_a^b |f(x) - h_n(x)|dx = 0$.

   (b) Let $\epsilon > 0$. Prove that there exists a step function $\psi : [a, b] \to \mathbb{R}$ such that

   $$\int_a^b |f(x) - \psi(x)|dx < \epsilon.$$

   (c) Let $\epsilon > 0$. Prove that there exists a continuous function $g : [a, b] \to \mathbb{R}$ such that $g(a) = g(b) = 0$ and

   $$\int_a^b |f(x) - g(x)|dx < \epsilon.$$

4. Suppose that $f : [-\pi, \pi] \to \mathbb{C}$ is Lebesgue integrable. The *Fourier coefficients* $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ of f are defined via

   $$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad n \in \mathbb{Z}.$$

   Prove the Riemann-Lebesgue lemma:

   $$\lim_{|n|\to\infty} |\hat{f}(n)| = 0. \tag{†}$$

   Hint: Prove (†) for step functions first. Then use problem 3 and an approximation argument.

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