Assignments are to be submitted to Gradescope by 24:00.

1. Let $f : \mathbb{R} \to \mathbb{R}$. Prove that the collection of sets

$$\mathcal{A} = \big\{E \subset \mathbb{R} \mid f^{-1}(E) \text{ is Lebesgue measurable}$$

is a σ-algebra.

2. Let $E \subset \mathbb{R}$, and assume that $m^*(E) < \infty$. Prove that E is measurable if and only if for every $\epsilon > 0$ there exists a finite union of open intervals U such that $m^*(U \Delta E) < \epsilon$. This result is known as *Littlewood's first principle*: every measurable set is nearly a finite union of open intervals.

   Hint: To prove the converse direction, let $A \subset \mathbb{R}$, and prove that for every $\epsilon > 0$,

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A) + \epsilon.$$

   You may use without proof the fact that a finite union of open intervals is measurable. This is covered in Lecture 8 which has been moved to Week 5.

3. Let E be a measurable set.

   (a) Prove that for all $x \in \mathbb{R}$, $E + x$ is measurable.

   (b) Prove that for all $r > 0$, $rE := \{ry \mid y \in E\}$ is measurable.

18.102 / 18.1021 Introduction to Functional Analysis
Spring 2021