Assignments are to be submitted Gradescope by 24:00.

1. Let $B$ be a Banach space.
   (a) Prove that if $T \in \mathcal{B}(B,B)$ and $\|I-T\| < 1$ where $I$ is the identity operator, then $T$ is invertible and in fact $\sum_{n=0}^{\infty}(I-T)^n$ converges in $\mathcal{B}(B,B)$ to $T^{-1}$.
   (b) Prove that the set of invertible operators is open in $\mathcal{B}(B,B)$.

2. Let $V$ be a normed vector space and $W \subset V$ a proper closed subspace.
   (a) Prove that $\|v+W\| := \inf_{w\in W}\|v+w\|$ is a norm on $V/W$.
   (b) Prove that for any $\epsilon > 0$ there exists $v \in V$ such that $\|v\| = 1$ and $\|v+W\| \geq 1-\epsilon$. *Hint:* Let $u \in V\backslash W$. Then $\|u+W\| > 0$ and there exists $w \in W$ such that $\|u+W\| \leq \|u+w\|$ and

   $$\|u+w\| \leq \|u+W\| + \epsilon\|u+W\|.$$

   Now consider $\frac{u+w}{\|u+w\|}$.

3. Let $V$ be a Banach space and $W \subset V$ a proper closed subspace. Prove that $V/W$ with the norm defined in problem 2 is a Banach space.

   *Hint:* Suppose that the series $\sum_n(v_n+W)$ is absolutely summable, i.e. $\sum_n\|v_n+W\|$ converges. We wish to prove that $\sum_n(v_n+W)$ converges in $V/W$. For each $n \in \mathbb{N}$, there exists $w_n \in W$ such that

   $$\|v_n+w_n\| \leq \|v_n+W\| + 2^{-n}.$$

   Then $\sum_n(v_n+w_n)$ is absolutely summable, and since $V$ is a Banach space, there exists $v \in V$ such that $v = \sum_n(v_n+w_n)$. Prove that $v+W = \sum_n(v_n+W)$, i.e.

   $$\lim_{N\to\infty} \left\| v+W - \sum_{n=1}^{N}(v_n+W) \right\| = 0.$$

4. Suppose $V$ and $W$ are Banach spaces, $T \in \mathcal{B}(V,W)$ and recall the following subspaces

   $$\ker(T) = \{v \in V \mid Tv = 0\}, \quad \text{range}(T) = \{Tv \in W \mid v \in V\}.$$

   (a) Prove that $\ker(T)$ is a closed subspace of $V$.
   (b) If $V_1$ and $V_2$ are normed linear spaces, we say a bijective linear operator $S : V_1 \to V_1$ is an *isomorphism* if $S \in \mathcal{B}(V_1,V_2)$ and $S^{-1} \in \mathcal{B}(V_2,V_1)$. We say $V_1$ and $V_2$ are *isomorphic* if there exists an isomorphism $S : V_1 \to V_2$.
   Prove that $V/\ker(T)$ is isomorphic to $\text{range}(T)$ if and only if $\text{range}(T)$ is closed.
   *Hint*: Consider the map $S : V/\ker T \to \text{range}(T)$ given by

   $$S(v+\ker T) = Tv,$$

   and first show that $S$ is a well-defined, bijective bounded linear operator.

5. The following exercise shows we cannot drop certain hypotheses in the closed graph theorem and open mapping theorem. Let

$$W = \left\{ a = \{a_k\}_k \mid \sum_k k|a_k| < \infty \right\},$$

equipped with the $\ell^1$ norm.

   (a) Prove that $W$ is a proper, dense subspace of $\ell^1$ (hence, $W$ is not complete).
   *Hint*: Show that if $b = \{b_k\}_k \in \ell^1$ and $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that if

   $$a := \{b_1, b_2, \ldots, b_N, 0, 0, \ldots\} \in W,$$

   then $\|a - b\|_1 < \epsilon$.

   (b) Define $T : W \to \ell^1$ by $(Ta)_k = ka_k$. Prove that the graph of $T$ is closed but $T$ is not bounded.

   (c) Let $S = T^{-1} : \ell^1 \to W$. Prove that $S$ is bounded and surjective but is not an open mapping.

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