Assignments are to be submitted to Gradescope by 24:00.
The following theorem (typically, but not always, covered in 18.100) will be useful.

Theorem (Arzela-Ascoli)**.** *Let $f_n \in C([a,b])$, $n \in \mathbb{N}$, such that*

- *there exists $B \geq 0$ such that for all $n \in \mathbb{N}$, $\|f_n\|_\infty \leq B$,*
- *the sequence $\{f_n\}_n$ is* equi-continuous*: for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|x-y| < \delta$, then for all $n \in \mathbb{N}$, $|f_n(x) - f_n(y)| < \epsilon$.*

Then there exists a subsequence $\{f_{n_j}\}_j$ converging in $C([a,b])$.

1. Let $\{f_n\}_n$ be a sequence of continuously differentiable functions on $[0,1]$ such that

$$B := \sup_n \Big[\|f_n\|_2 + \|f_n'\|_2 \Big] < \infty.$$

   (a) Prove that for all $n \in \mathbb{N}$, $\|f_n\|_\infty \leq B$.
   Hint: Prove for all $x, y \in [0,1]$,

$$|f_n(x)| \leq |f_n(y)| + \|f_n'\|_2.$$

   Now integrate in $y \in [0,1]$.

   (b) Prove that for all $n \in \mathbb{N}$ and $x, y \in [0,1]$, $|f_n(x) - f_n(y)| \leq B|x-y|^{1/2}$.

   (c) Prove that there exists a subsequence $\{f_{n_j}\}_j$ converging in $L^2([0,1])$.

2. Let $K \in C([a,b] \times [a,b])$ and define

$$Tf(x) = \int_a^b K(x,y) f(y) dy, \quad f \in L^2([a,b]).$$

   Prove that $T \in \mathcal{B}(L^2([a,b]))$, and for every bounded sequence $\{f_n\}_n$ in $L^2([a,b])$, the sequence $\{Tf_n\}_n$ has a subsequence $\{Tf_{n_j}\}_j$ converging in $L^2([a,b])$. [This proves that T is a compact operator.]

3. Recall, from Assignment 8, the Sobolev space $H^s(\mathbb{T})$ of order $s \geq 0$:

$$H^s(\mathbb{T}) := \{ f \in L^2([-\pi,\pi]) \mid \|f\|_{H^s(\mathbb{T})} < \infty \ ,$$
$$\|f\|_{H^s(\mathbb{T})}^2 := \sum_{n \in \mathbb{Z}} (1 + |n|^2)^s |\hat{f}(n)|^2.$$

   Prove that if $s > 0$ and $B \geq 0$, the set

$$K = \{ f \in H^s(\mathbb{T}) \mid \|f\|_{H^s(\mathbb{T})} \leq B \},$$

   is a compact subset of $L^2([-\pi,\pi])$. [This proves that the inclusion $\iota : H^s(\mathbb{T}) \to L^2(\mathbb{T})$ is a compact operator when $s > 0$.]

4. Let $\{A_{ij}\}_{i,j=1}^{\infty}$ be a bi-sequence of complex numbers such that

$$\sum_i \sum_j |A_{ij}|^2 := \lim_{N\to\infty} \sum_{i=1}^{N} \Big(\sum_{j=1}^{\infty} |A_{ij}|^2 \Big) < \infty.$$

For $a = \{a_j\}_j \in \ell^2$, define a sequence $Ta = \{(Ta)_i\}_i$ by

$$(Ta)_i := \sum_j A_{ij} a_j.$$

(a) Prove that $T \in \mathcal{B}(\ell^2)$ and $\|T\| \le \Big(\sum_i \sum_j |A_{ij}|^2 \Big)^{1/2}$.

(b) For $n \in \mathbb{N}$, define the n-th truncation of T via

$$(T_n a)_i = \begin{cases} \sum_{j=1}^n A_{ij} a_j & \text{if } i = 1, 2, \ldots, n \\ 0 & \text{if } i = n+1, n+2, \ldots, \end{cases} \qquad a \in \ell^2.$$

Prove that $T_n \in \mathcal{B}(\ell^2)$ is a finite rank operator and $\lim_{n\to\infty} \|T - T_n\| = 0$. [According to a theorem proved in class, this shows that T is a compact operator.]

5. For $a = \{a_k\} \in \ell^2$, we define the *left shift operator*

$$La = \{a_2, a_3, a_4, a_5, \ldots\},$$

and the *right shift operator*

$$Ra = \{0, a_1, a_2, a_3, \ldots\}.$$

(a) Is L or R a compact operator? Why or why not?

(b) Prove that R has no eigenvalues.

(c) Prove that $\mathrm{Spec}(R) = \{\lambda \in \mathbb{C} \mid |\lambda| \le 1\}$.

(d) Prove that the set of eigenvalues of L is given by $\{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$, and determine the corresponding eigenspaces.
Hint: As evident from the definitions, $La = \lambda a$ if and only if for all $k \in \mathbb{N}$, $a_{k+1} = \lambda a_k$. Now solve this recursive equation.

(e) Prove that $\mathrm{Spec}(L) = \{\lambda \in \mathbb{C} \mid |\lambda| \le 1\}$.

6. (Do not turn in.) Given $f \in L^2([0,1])$, define

$$Tf(x) = \int_0^x f(t)dt, \quad x \in [0,1].$$

(a) Prove that $T \in \mathcal{B}(L^2([0,1]))$ and T is a compact operator.
Hint: Use Cauchy-Schwarz to prove that

$$\forall x \in [0,1], \quad |Tf(x)| \le \|f\|_2,$$
$$\forall x, y \in [0,1], \quad |Tf(x) - Tf(y)| \le \|f\|_2 |x - y|^{1/2}.$$

These estimates alone suffice to prove that $T \in \mathcal{B}(L^2([0,1]))$, and in conjunction with Arzela-Ascoli they can be used to prove T is a compact operator.

(b) In the next two parts, we will show that T has no eigenvalues. Suppose that $f \in L^2([0,1])$ and $Tf = 0$. Prove that for all $a, b \in [0,1]$ with $a < b$, we have $\int_a^b f(t)dt = 0$. Conclude that $\int_0^1 f(t)\overline{\chi(t)}dt = 0$ for all simple functions in $L^2([0,1])$, and thus, $f = 0$. [Thus, T is injective, and zero is not an eigenvalue of T.]

(c) Suppose that $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Prove that if $(T - \lambda I)f = 0$, then f is continuously differentiable and

$$f - \lambda f' = 0, \quad f(0) = 0.$$

Conclude that $f = 0$. [Thus, $T - \lambda I$ is injective, and λ is not an eigenvalue of T.]

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