18.085 Computational Science and Engineering I
Fall 2008

18.085: Matlab Homework #5

Laplace's equation.

The problem is Laplace's equation on the unit square with boundary conditions $u = y$ on the side $x = 1$, $u = x$ on the side $y = 1$, $w \cdot n = -y$ on the side $x = 0$, $w : n = -x$ on the side $y = 0$.

Replace the 2nd derivatives in Laplace's equation by centered second differences. This gives the "5-point discrete Laplacian" on a square grid ($\Delta x = \Delta y$):

$$[u(x + \Delta x, y) + u(x - \Delta x, y) + u(x, y + \Delta x) + u(x, y - \Delta x) - 4u(x, y)]/(\Delta x)^2$$

Set $\Delta x = 1/11$ giving $10 \times 10 = 100$ interior grid points with 100 unknowns $u(x, y)$. The 100 grid points fall in a square array but you have to make them into a VECTOR with 100 components. I usually number them by rows, $u_1 = u(\Delta x, \Delta x)$ and $u_2 = u(\Delta x, 2\Delta x)$ and next row $u_{11} = u(2\Delta x, \Delta x)$ and last corner $u_{100} = u(10\Delta x, 10\Delta x)$.

At two boundaries we know u. At the left boundary $x = 0$ we know $u' = -y$. Replace by $[u(\Delta x, y) - u(0, y)]/\Delta x = -y$. This gives $u(0, y)$ in terms of $u(\Delta x, \Delta x)$; Substitute in the 5-point equation to eliminate $u(0, y)$. Similarly, eliminate $u(x, 0)$ on the boundary $y = 0$ where $\partial u/\partial y = -x$.

Set up the whole system as $Ku = f$ where K is 100×100. Is K symmetric? Is K positive definite?? (Let MATLAB decide.) Print out diag(K). IT SHOULD NOT BE ALL 4's.

K is banded around the main diagonal. What is the bandwidth so $K(i, j) = 0$ if abs $(i - j) > w$? What are the largest and smallest eigenvalues of K? Use the command eig(K).

Print out v, the first column or row of K inverse. This gives the value of $U(\Delta x, \Delta x)$ at the lower left corner from the right side f. What is the ratio of the first component $v(1, 1)$ to the last component $v(10, 10)$?

Bonus: Let Matlab solve $Ku = 0$ and plot $u(x, y)$ (Remember, $u(x, y)$ is a 2D surface). Matlab also has a plotting command called QUIVER which plots velocity fields. If $u(x, y)$ describes potential flow, there is a velocity field associated with u given by $v = \nabla u$. Plot this velocity field.