Further Numerical Methods

Euler's method is a *first order method* (no relation to *first order equations*). It is possible to show theoretically that for small enough h, the error in Euler's method is at most C_1h, where C_1 is a constant that depends on the IVP. It is very hard to know ahead of time what C_1 will be.

In the previous section, we saw that making h smaller was a way to decrease the error caused by the variability of the direction field. However, there are some more sophisticated methods that turn out to be even better.

1. General Approach

Looked at broadly Euler's method is a way of stepping discretely from one point to the next to approximate the integral curve. The general formula for stepping from (x_n, y_n) to (x_{n+1}, y_{n+1}) is

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + mh,$$

where h is the stepsize in the x direction and m is the slope of the line we step along. In Euler's method h is fixed ahead of time and $m = f(x_n, y_n)$. (It would be more precise to write m_n instead of m. We'll use the simpler looking notation, with the understanding that m changes with each step.)

Other methods use other (and better) ways of choosing h and m. We start with some *fixed stepsize* methods. As the name suggests, we fix the stepsize h ahead of time and put all the work into finding m

2. The Improved Euler method

This is also called the *Runge-Kutta 2* method or *RK2*, or the *Heun* method.

We start with the same data as for Euler's method: an initial value problem $y' = f(x, y)$, $y(x_0) = x_0$, and a step size h. We construct an RK2 polygon, made out of segments called RK2 struts, with endpoints (x_n, y_n). As before, $x_{n+1} = x_n + h$. The difference between RK2 and Euler's method is the rule for choosing the slope m for each strut. At each step we start by constructing the Euler strut. We let m be the average of the slope field at the two ends of the strut.

Example. Consider the differential equation $y' = f(x, y) = y^2 - x$ with initial condition $y(0) = -1$. Let us compute one step for the RK2 polygon with $h = 1/2$.
Because m, x and y are reserved we'll use the letters k, a and b for intermediate slopes and points.)

1. Compute the slope at (x_0, y_0): $k_1 = f(0, -1) = 1$.
2. Take an Euler step from (x_0, y_0) to (a, b): $a = x_0 + h = .5$, $b = y_0 + k_1 h = -.5$.
3. Compute the slope at (a, b): $k_2 = f(a, b) = f(.5, -.5) = -.25$.
4. Average k_1 and k_2 to get m: $m = (k_1 + k_2)/2 = .375$. 5. Use m and h to take step from (x_0, y_0) to (x_1, y_1): $x_1 = x_0 + h = .5$, $y_1 = y_0 + mh = -.8125$.

You can check, e.g. by using the applet, that this brings us down closer to the actual solution curve than Euler's method.

RK2 is a *second order method*: for small enough h, the error is at most $C_2 h^2$, where the constant C_2 depends on the IVP.

Each evaluation of the direction field takes time, which usually costs money. Euler's method uses one evaluation per step, whereas RK2 uses two; therefore, if we want to compare efficiencies, we should compare Euler's method with step size h to RK2 with $2h$. In those cases, the error for Euler's method is around $C_1 h$, whereas it is around $C_2(2h)^2 = 4C_2 h^2$ for RK2. Even if C_2 is larger than C_1, for small enough h, the RK2 error will be significantly smaller than the Euler error. Besides, C_2 is usually smaller than C_1, which gives a second advantage to using RK2 over Euler's method.

3. Runge-Kutta 4 method

This is usually shortened to *RK4*. It is a refinement of RK2; we start with the same data, and also build a polygon, whose segments are called RK4 struts. Again, at each step, the difference is in choosing the slope of the segment.

In RK4 you evaluate the direction field slope four times for each step. We won't give the details, they are easy enough to look up.

Remark 1. While it's straightforward to compute by hand, most people leave the computations in RK4 to a computer.

Remark 2. You might have noticed a pattern in the numbering of the Runge-Kutta techniques; Euler's method is sometimes referred to as RK1.

RK4 is a *fourth order method*. For small enough h, its error is approximately $C_4 h^4$. Again, the constant C_4 depends on the IVP.

It is fair to compare the errors for Euler's method with step size h, RK2 with step size $2h$, and RK4 with $4h$. Regardless of the values of C_1, C_2 and C_4, for sufficiently small h, the RK4 error of $C_4(4h)^4$ will be significantly less than the RK2 error of $C_2(2h)^2$ or the Euler error of C_1h. Besides, C_4 itself is usually smaller than C_2 and C_1.

Example. Let us go back to our original problem: estimating e by viewing it as the value at 1 of the solution to the initial value problem $y' = y, y(0) = 1$. We compare the errors of our three methods. In all cases, we use 1000 evaluations of the direction field.

Method	Step size	Error
RK1 = Euler	0.001	1.3×10^{-3}
RK2 = Heun	0.002	1.8×10^{-6}
RK4	0.004	5.8×10^{-12}

We can also estimate the constants C_i for this particular IVP: C_1 1.3; C_2 0.45; C_4 0.023.

The (short) moral is that Euler's method often offers poor precision, and that RK4 is essentially always the most accurate.

As you might have guessed, there are plenty of methods of higher order still; however, they also involve more overhead. Experience has shown that RK4 is a good compromise.

Remark. The initial value problem $y' = f(x)$, $y(a) = y_0$ has solution $y(x) = y_0 + \int_a^x f(t)dt$ Our numerical methods for approximating y(x) correspond to integration approximation techniques:

- Euler's method gives the the left end-point Riemann sum;
- RK2 gives the trapezoidal rule;
- RK4 gives Simpson's rule.

4. Variable Stepsize methods

We saw that it is not wise to pick a single stepsize and accept the results of the Euler method. Likewise RK2 and RK4 can be fooled.

With the fixed stepsize methods you need to choose a value of h; do your computation; then redo it using stepsize $h/2$. You keep cutting the stepsize in half this until the answers stop changing. The variable stepsize techniques carry this out at each step. There are an enormous number of such methods. What they have in common is estimating at each step whether the stepsize needs to be made smaller or can safely be made larger. In general, these provide the most accurate numerical methods at an acceptable cost in additional computation.

18.03SC Differential Equations
Fall 2011