18.01 Single Variable Calculus
Fall 2006

# Lecture 22: Volumes by Disks and Shells

## Disks and Shells

We will illustrate the 2 methods of finding volume through an example.

**Example 1. A witch's cauldron**

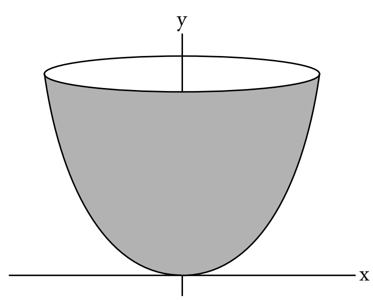


Figure 1: $y = x^2$ rotated around the $y$-axis.

### Method 1: Disks

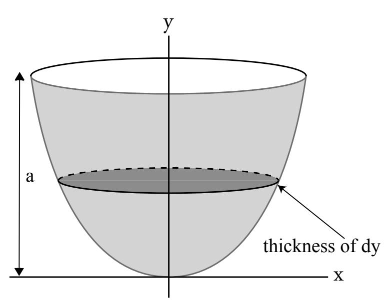


Figure 2: Volume by Disks for the Witch's Cauldron problem.

The area of the disk in Figure 2 is $\pi x^2$. The disk has thickness $dy$ and volume $dV = \pi x^2 dy$. The volume $V$ of the cauldron is

$$V = \int_0^a \pi x^2\, dy \quad \text{(substitute} \quad y = x^2)$$

$$V = \int_0^a \pi y\, dy = \pi \frac{y^2}{2}\Big|_0^a = \frac{\pi a^2}{2}$$

If $a = 1$ meter, then $V = \frac{\pi}{2}a^2$ gives

$$V = \frac{\pi}{2}\ m^3 = \frac{\pi}{2}(100\text{ cm})^3 = \frac{\pi}{2}10^6\text{ cm}^3 \approx 1600\text{ liters} \quad \text{(a huge cauldron)}$$

**Warning about units.**

If $a = 100\,\text{cm}$, then

$$V = \frac{\pi}{2}(100)^2 = \frac{\pi}{2}10^4\text{ cm}^3 = \frac{\pi}{2}10 \sim 16\text{ liters}$$

But 100cm = 1m. Why is this answer different? The resolution of this paradox is hiding in the equation.

$$y = x^2$$

At the top, $100 = x^2 \implies x = 10\,\text{cm}$. So the second cauldron looks like Figure 3. By contrast, when

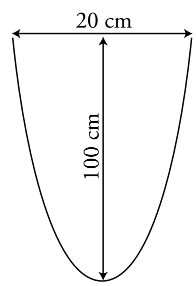


Figure 3: The skinny cauldron.

$a = 1\,\text{m}$, the top is ten times wider: $1 = x^2$ or $x = 1\,\text{m}$. Our equation, $y = x^2$, is not scale-invariant. The shape described depends on the units used.

## Method 2: Shells

This really should be called the cylinder method.

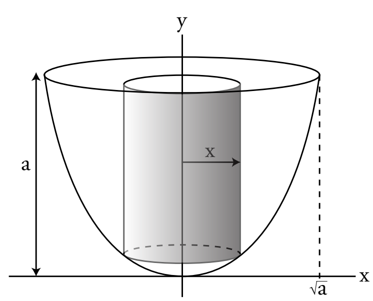


Figure 4: $x$ = radius of cylinder. Thickness of cylinder = $dx$. Height of cylinder = $a - y = a - x^2$.

The thin shell/cylinder has height $a - x^2$, circumference $2\pi x$, and thickness $dx$.

$$\begin{aligned}
dV &= (a - x^2)(2\pi x)dx \\
V &= \int_{x=0}^{x=\sqrt{a}} (a - x^2)(2\pi x)dx = 2\pi \int_0^{\sqrt{a}} (ax - x^3)dx \\
&= 2\pi \left(a\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^{\sqrt{a}} = 2\pi \left(\frac{a^2}{2} - \frac{a^2}{4}\right) = 2\pi \left(\frac{a^2}{4}\right) = \frac{\pi a^2}{2} \quad \text{(same as before)}
\end{aligned}$$

**Example 2. The boiling cauldron**
Now, let's fill this cauldron with water, and light a fire under it to get the water to boil (at $100^o$C). Let's say it's a cold day: the temperature of the air outside the cauldron is $0^o$C. How much energy does it take to boil this water, i.e. to raise the water's temperature from $0^o$C to $100^o$C? Assume the

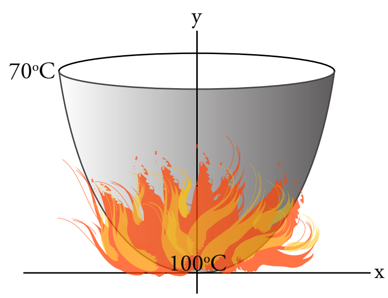


Figure 5: The boiling cauldron ($y = a = 1$ meter.)

temperature decreases linearly between the top and the bottom ($y = 0$) of the cauldron:

$$T = 100 - 30y \quad \text{(degrees Celsius)}$$

Use the method of disks, because the water's temperature is constant over each horizontal disk. The total heat required is

$$\begin{aligned}
H &= \int_0^1 T(\pi x^2)dy \quad \text{(units are (degree)(cubic meters))} \\
&= \int_0^1 (100 - 30y)(\pi y)dy \\
&= \pi \int_0^1 (100y - 30y^2)dy = \pi(50y^2 - 10y^3)\Big|_0^1 = 40\pi \,(\text{deg.})\text{m}^3
\end{aligned}$$

How many calories is that?

$$\#\text{ of calories } = \frac{1\,\text{cal}}{\text{cm}^3 \cdot \text{deg}}(40\pi)\left(\frac{100\,\text{cm}}{1\,\text{m}}\right)^3 = (40\pi)(10^6)\,\text{cal} = 125 \times 10^3\,\text{kcal}$$

There are about 250 kcals in a candy bar, so there are about

$$\#\text{ of calories } = \left(\frac{1}{2}\text{candy bar}\right) \times 10^3 \approx 500\,\text{candy bars}$$

So, it takes about 500 candy bars' worth of energy to boil the water.

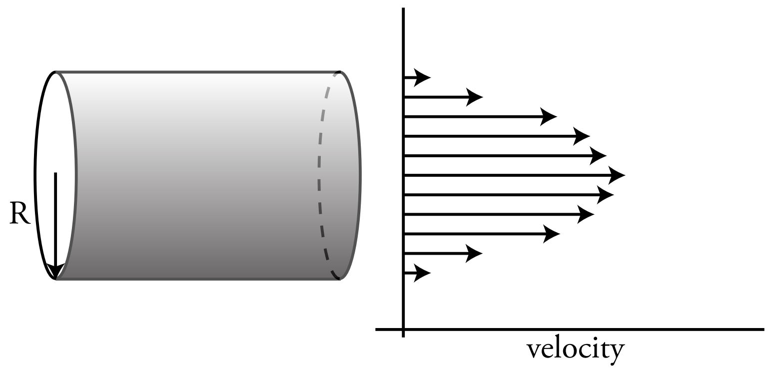


Figure 6: Flow is faster in the center of the pipe. It slows– "sticks"– at the edges (i.e. the inner surface of the pipe.)

**Example 3. Pipe flow**
Poiseuille was the first person to study fluid flow in pipes (arteries, capillaries). He figured out the velocity profile for fluid flowing in pipes is:

$$v = c(R^2 - r^2)$$
$$v = \text{speed} = \frac{\text{distance}}{\text{time}}$$

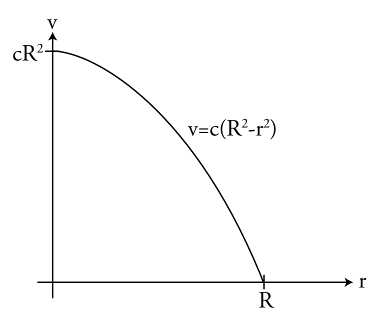


Figure 7: The velocity of fluid flow vs. distance from the center of a pipe of radius $R$.

The flow through the "annulus" (a.k.a ring) is (area of ring)(flow rate)

$$\text{area of ring} = 2\pi r dr \quad \text{(See Fig. 8: circumference } 2\pi r\text{, thickness } dr\text{)}$$

$v$ is analogous to the height of the shell.

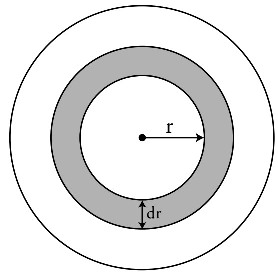


Figure 8: Cross-section of the pipe.

$$\begin{aligned}
\text{total flow through pipe} &= \int_0^R v(2\pi r dr) = c\int_0^R (R^2 - r^2) 2\pi r dr \\
&= 2\pi c \int_0^R (R^2 r - r^3) dr = 2\pi c \left(\frac{R^2 r^2}{2} - \frac{r^4}{4}\right)\Big|_0^R \\
\text{flow through pipe} &= \frac{\pi}{2} c R^4
\end{aligned}$$

Notice that the flow is proportional to $R^4$. This means there's a big advantage to having thick pipes.

**Example 4. Dart board**
You aim for the center of the board, but your aim's not always perfect. Your number of hits, $N$, at radius $r$ is proportional to $e^{-r^2}$.

$$N = ce^{-r^2}$$

This looks like:

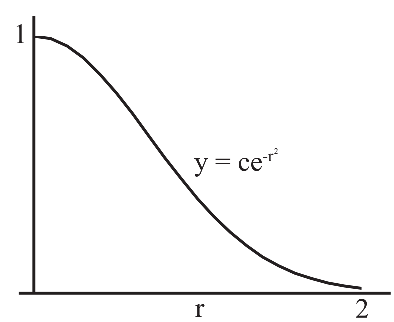


Figure 9: This graph shows how likely you are to hit the dart board at some distance $r$ from its center.

The number of hits within a given ring with $r_1 < r < r_2$ is

$$c\int_{r_1}^{r_2} e^{-r^2}(2\pi r dr)$$

We will examine this problem more in the next lecture.