18.01 Single Variable Calculus
Fall 2006

Practice Final SOLUTIONS 18.01

$\boxed{1}$ a) $\dfrac{x^2\cdot\frac{1}{x} - \ln x\cdot 2x}{x^4} = \dfrac{1-2\ln x}{x^3}$

b) $\frac{1}{2}(3\sin^2 u+2)^{-1/2}\cdot 6\sin u\cos u$

$= \dfrac{3\sin u\cos u}{\sqrt{3\sin^2 u+2}}$

c) $D^n e^{kx} = k^n e^x$; at 0: $\boxed{k^n}$

$\boxed{2}$ $D_x(x^2y^2+y^3) = 2xy^2 + x^2\cdot 2yy' + 3y^2y'$
$= 0.$

At $(1,1)$: $2+2y'+3y'=0$; $y' = -\frac{2}{5}$

Eq'n: $(y-1) = -\frac{2}{5}(x-1)$ or

$y = -\frac{2}{5}x + \frac{2}{5}$

$\boxed{3}$ $y = \cos^{-1} x$

$x = \cos y$

$1 = -\sin y\cdot y'$

$y' = \dfrac{-1}{\sin y}$

$= \dfrac{-1}{\sqrt{1-\cos^2 y}}$

$\therefore\ y' = \dfrac{-1}{\sqrt{1-x^2}}$

(use $+\sqrt{\ }$ since $\sin y > 0$)

$\boxed{4}$ $\begin{cases} x^2+x+a \\ bx+2\end{cases}$ same value at 0 (for continuity)

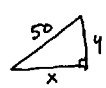
$\Rightarrow \boxed{a=2}$

same deriv. at 0 (for diff'y)

$\Rightarrow 2x+1 = b$ at 0

$\therefore \boxed{b=1}$

$\boxed{5}$



when $y=30$, $x=40$

$x^2+y^2=50^2$

$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = 0$

$40\cdot\frac{dx}{dt} + 30(-2) = 0$

$\frac{dx}{dt} = \frac{60}{40} = \boxed{\frac{3}{2}}$ ft/sec

$\boxed{6}$ $A = 2x\cdot(1-x^2) = 2x-2x^3$

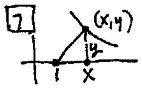
$\frac{dA}{dt} = 2-6x_1^2 = 0$ if $x^2 = \frac{1}{3}$

$x = \frac{1}{\sqrt{3}}$

Area then is:

$2\cdot\frac{1}{\sqrt{3}}\left(1-\frac{1}{3}\right) = \boxed{\dfrac{4}{3\sqrt{3}}}$

$\boxed{7}$



$-\dfrac{1}{y'} = \dfrac{y}{x-1}$

(negative recip. slope)

a) $\therefore\ y' = \dfrac{1-x}{y}$

b) $\frac{dy}{dx} = \frac{1-x}{y}$

$y\,dy = (1-x)\,dx$

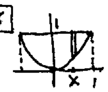
$\frac{1}{2}y^2 = x - \frac{1}{2}x^2 + c_1$ $\left[\text{or better: } -\frac{1}{2}(1-x)^2 + c_2\right]$

$y^2 = 2x - x^2 + c$

c) $y^2 + (x-1)^2 = c_3$ (completing square)

circles centered at $(1,0)$

$\boxed{8}$



Volume $= \int_0^1 2\pi x(1-x^2)\,dx$

$= 2\pi\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big]_0^1$

$= 2\pi\cdot\frac{1}{4} = \boxed{\frac{\pi}{2}}$

[or: $\underbrace{\pi\cdot 1^2\cdot 1}_{\text{vol. cylinder}} - \underbrace{\int_0^1 2\pi x^3\,dx}_{\text{vol. under curve}} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$]

$\boxed{9}$

x	0	$\pi/6$	$\pi/3$	$\pi/2$
$\sin x$	0	$1/2$	$\sqrt{3}/2$	1
$\sin^2 x$	0	$1/4$	$3/4$	1

Trap. rule: $\frac{\pi}{6}\left(\frac{1}{2}\cdot 0 + \frac{1}{4} + \frac{3}{4} + \frac{1}{2}\cdot 1\right)$

$= \frac{\pi}{6}\cdot\frac{3}{2} = \boxed{\frac{\pi}{4}}$

$\boxed{10}$ $F(x) = \int_0^x e^{-t^2}\,dt$

a) $F'(x) = e^{-x^2}$; $F''(x) = -2xe^{-x^2}$

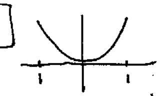
$F'(1) = \frac{1}{e}$ $\quad F''(1) = \frac{-2}{e}$

b) $\int_1^2 e^{-u^2/4}\,du = \int_{1/2}^1 e^{-t^2}\cdot 2\,dt$

put $t = u/2$, $dt = \frac{du}{2}$

$= 2(F(1) - F(1/2))$

$\boxed{11}$



$y = x^2/10$

$y' = x/5$

a) $\therefore$ arclength $= \int_{-1}^1 \sqrt{1+\frac{x^2}{25}}\,dx$

b) average $= \int_0^1 \frac{x^2}{10}\,dx = \frac{x^3}{30}\Big]_0^1$

$= \frac{1}{30}$ km ≈ 33 m

[or $= \frac{1}{2}\int_{-1}^1 \frac{x^2}{10}\,dx = \ldots$]

$\boxed{12}$ a) $\dfrac{1}{x^2+3x+2} = \dfrac{1}{(x+2)(x+1)} = \dfrac{-1}{x+2} + \dfrac{1}{x+1}$ (by coverup)

$\therefore \int_0^1\left(\frac{-1}{x+2} + \frac{1}{x+1}\right)dx = -\ln(x+2) + \ln(x+1)\Big]_0^1$

$= -\ln 3 + \ln 2 + \ln 2 - 0$

$= 2\ln 2 - \ln 3$ (or $\ln\frac{4}{3}$)

b) $\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \int\frac{x^3}{3}\cdot\frac{1}{x}\,dx$

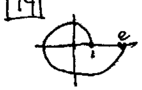
int. by parts $= \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$

$\boxed{13}$ $\int_0^1 \dfrac{dx}{(x^2+1)^2} = \int_0^{\pi/4}\dfrac{\sec^2 u\,du}{\sec^4 u} = \int_0^{\pi/4}\cos^2 u\,du$

$x = \tan u$ $\quad = \int_0^{\pi/4}\frac{1+\cos 2u}{2}\,du = \frac{u}{2} + \frac{\sin 2u}{4}\Big]_0^{\pi/4}$

$= \boxed{\frac{\pi}{8} + \frac{1}{4}}$

$\boxed{14}$



Area $= \frac{1}{2}\int_0^{2\pi}\left(e^{\theta/2\pi}\right)^2 d\theta$

$= \frac{1}{2}\int_0^{2\pi} e^{\theta/\pi}\,d\theta = \frac{\pi}{2}e^{\theta/\pi}\Big]_0^{2\pi}$

$= \frac{\pi}{2}(e^2-1)$

$\boxed{15}$ a) $\lim_{x\to 0}\dfrac{\sin^2 x}{1-\cos x} = \lim_{x\to 0}\dfrac{2\cancel{\sin x}\cos x}{\cancel{\sin x}} = 2$

b) $\lim_{x\to 1}\dfrac{(\ln x)^2}{x-1} = \lim_{x\to 1}\dfrac{2\ln x\cdot\frac{1}{x}}{1} = 0$

c) $\lim_{x\to\infty}\dfrac{x^2}{e^x} = \lim_{x\to\infty}\dfrac{2x}{e^x} = \lim_{x\to\infty}\dfrac{2}{e^x} = 0$

$\boxed{16}$ $\int_1^\infty\dfrac{dx}{x^{3/2}} = \int_1^\infty x^{-3/2}\,dx = -2x^{-1/2}\Big]_1^\infty = 0-(-2)$
$= 2$

$\boxed{17}$ $\dfrac{n}{\sqrt{4+n^p}} \sim \dfrac{n}{n^{p/2}} \sim \dfrac{1}{n^{p/2-1}}$

$\therefore \sum\frac{n}{\sqrt{4+n^p}}$ converges if $\frac{p}{2}-1 > 1$
or $\boxed{p>4}$

$\boxed{18}$

	at $x=0$:	coeff:
$y = (1+x)^{1/2}$	1	1
$y' = \frac{1}{2}(1+x)^{-1/2}$	$1/2$	$1/2$
$y'' = -\frac{1}{2}\cdot\frac{1}{2}(1+x)^{-3/2}$	$-1/4$	$-1/4\cdot\frac{1}{2!}$
$y''' = -\frac{3}{2}\cdot -\frac{1}{2}\cdot\frac{1}{2}(1+x)^{-5/2}$	$3/8$	$3/8\cdot\frac{1}{3!}$

$\therefore (1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

$(1.2)^{1/2} = 1 + \frac{1}{2}\cdot\frac{2}{10} - \frac{1}{8}\cdot\frac{4}{100} + \frac{1}{16}\cdot\frac{8}{1000}$

$= 1 + .1000 - .005 + .0005$

$= \boxed{1.0955}$

$\boxed{19}$ $y = \tan^{-1}x$

$y' = \dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8\ldots$

Integrate term by term $\therefore\ y = \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}\ldots + c$

$c=0$ since $\tan^{-1}0 = 0$.