18.01 Single Variable Calculus
Fall 2006

18.01 Exam 2

Tuesday, Oct. 17, 2006

Name: SOLUTIONS

Put a check next to your recitation:

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Rec. 7	Matthew Hedden	2pm	
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Problem 1	15 points	
Problem 2	20 points	
Problem 3	20 points	
Problem 4	15 points	
Problem 5	20 points	
Problem 6	10 points	

Problem 1. (15 pts). Estimate the following to two decimal places (show work)

a. (8 pts). $\sin(\pi + 1/100)$

$$\sin\left(\pi+\tfrac{1}{100}\right) \approx \sin(\pi) + \cos(\pi)\tfrac{1}{100}$$
$$= 0 - 1\tfrac{1}{100} = -0.01$$

b. (7 pts). $\sqrt{101}$.

$$\sqrt{101} = \sqrt{100+1} = \sqrt{100(1+\tfrac{1}{100})} = 10\sqrt{1+\tfrac{1}{100}}$$
$$\approx 10\left(1+\tfrac{1}{2}\tfrac{1}{100}\right) = 10 + \tfrac{1}{20} = 10.05$$

Problem 2. (20 pts). Sketch the graph of $y = \frac{4}{x} + x + 1$ on $-\infty < x < \infty$ and label all critical points and inflection points with their coordinates on the graph along with the letter "C" or "I".

Note $f(x)$ is not defined at $x=0$

as $x \to \infty$ $\quad y \approx x+1 \to \infty$

as $x \to -\infty$ $\quad y \approx x+1 \to -\infty$

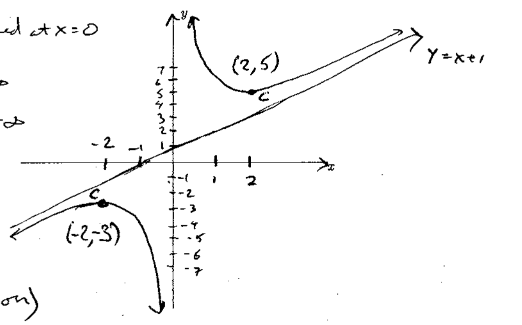
$y' = -\frac{4}{x^2} + 1$

$y' = 0 \iff x = \pm 2$ (critical points)

$y'' = 8/x^3$ (no inflection points)

$f''(2) = 1 > 0$

$f''(-2) = -1 < 0$



Write down on which intervals the function is:

increasing: $y' > 0$

$-\infty < x \le 2$

$2 \le x < \infty$

decreasing: $y' < 0$

$-2 \le x < 0$; $0 < x \le 2$ — NOT $-2 < x < 2$ because the function is undefined at $x=0$.

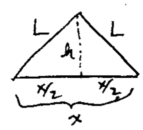
concave up: $y'' > 0$

$0 < x < \infty$

concave down: $y'' < 0$

$-\infty < x < 0$

Problem 3. (20 pts.) An architect plans to build a triangular enclosure with a fence on two sides and a wall on the third side. Each of the fence segments has fixed length L. What is the length x of the third side if the region enclosed has the largest possible area? Show work and include an argument to show that your answer really gives the maximum area.



$A = \frac{1}{2} x h$, $\quad h^2 + (x/2)^2 = L^2$, $\quad 0 \le x \le 2L$.

METHOD 1 (Substitution) $\quad h = \sqrt{L^2 - x^2/4}$, $\quad A = \frac{1}{2} x \sqrt{L^2 - x^2/4}$

$A' = \frac{1}{2}\left(\sqrt{L^2 - x^2/4} + \frac{1}{2} x \left(\frac{-2x}{4}\right)/\sqrt{L^2 - x^2/4}\right) = \frac{1}{2\sqrt{L^2 - x^2/4}}\left(L^2 - \frac{x^2}{4} - \frac{x^2}{4}\right) = 0$

Hence $L^2 = x^2/2$, and $\boxed{x = \sqrt{2}L}$

MAX BECAUSE: $A = 0$ at both ends $x=0$ and $x=2L$ and $A>0$ at the unique critical pt. in between. So this crit pt must be where max is achieved. (2nd deriv test is longer, but O.K., too.)

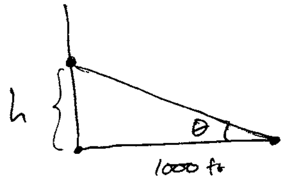
METHOD 2. (Implicit Diff.) no square roots until the end. $\quad 2hh' + x/2 = 0 \Rightarrow h' = \frac{-x}{4h}$

$0 = A' = \frac{1}{2}(h + xh') = \frac{1}{2}\left(h + x\left(\frac{-x}{4h}\right)\right) \Rightarrow h = x^2/4h$

$\Rightarrow h^2 = x^2/4 \Rightarrow \frac{x^2}{4} + \frac{x^2}{4} = L^2 \Rightarrow x^2 = 2L^2 \Rightarrow \boxed{x = \sqrt{2}L}$

(REASONING FOR MAX IS THE SAME.)

Problem 4. (15 pts.) A rocket is launched straight up, and its altitude is $h = 10t^2$ feet after t seconds. You are on the ground 1000 feet from the launch site. The line of sight from you to the rocket makes an angle θ with the horizontal. By how many radians per second is θ changing ten seconds after the launch?



$$\tan\theta = h/1000 = \frac{10t^2}{1000} = \frac{1}{100}\cdot t^2$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{50}t$$

when $t = 10$, $h(10) = 10\cdot 10^2 = 1000$, so

$\tan\theta = 1$ so $\theta = \pi/4$

$$\frac{1}{\cos^2(\frac{\pi}{4})}\left.\frac{d\theta}{dt}\right|_{t=10} = \frac{1}{50}\cdot 10$$

$$\left.\frac{d\theta}{dt}\right|_{t=10} = \frac{1}{5}\cos^2\frac{\pi}{4} = \frac{1}{5}\left(\frac{\sqrt{2}}{2}\right)^2 = \boxed{\frac{1}{10}\ \text{rad}/\text{s}}$$

Problem 5. a. (10 pts) Evaluate the following indefinite integrals

i. $\int \cos(3x)\,dx = \frac{1}{3}\sin(3x) + C$

ii. $\int xe^{(x^2)}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C$

$u = x^2$

$du = 2x\,dx$

$= \frac{1}{2}e^{(x^2)} + C$

b. (10 pts) Find $y(x)$ such that $y' = \frac{1}{y^3}$ and $y(0) = 1$

$$\frac{dy}{dx} = \frac{1}{y^3}$$

$$y^3\,dy = dx$$

$$\frac{1}{4}y^4 = x + C$$

$$\frac{1}{4} = C$$

$$y = (4x+1)^{1/4}$$

Problem 6. (10 pts.) Suppose that $f'(x) = e^{(x^2)}$, and $f(0) = 10$. One can conclude from the mean value theorem that

$$A < f(1) < B$$

for which numbers A and B?

$\frac{f(1) - f(0)}{1-0} = f'(c)$ for some c, $0 < c < 1$.

Or $f(1) = f(0) + f'(c) = 10 + e^{(c^2)}$, $0 < c < 1$.

If $0 < c < 1$, then $e^{(c^2)} < e$ and $1 < e^{(c^2)}$.

So $f(1) = 10 + e^{(c^2)} < 10 + e$

and $f(1) = 10 + e^{(c^2)} > 10 + 1 = 11$

$$\boxed{\begin{array}{l} A = 11 \\ B = 10 + e \end{array}}$$