Ordinals as Blueprints

1 Ordinal Precedence v. Cardinal Precedence

We have discussed two different precedence relations, $<_0$ and $<$:

- $<_o$ is the precedence relation for ordinals.

  $\alpha <_o \beta$ means that α precedes β in the hierarchy of ordinals.

- $<$ is an ordering of set-cardinality.

  $|A| < |B|$ means that there is an injection from A to B (but no bijection).

> *Important:* $\alpha <_o \beta$ does not entail $|\alpha| < |\beta|$.

2 Ordinals as Blueprints for Large Sets

- An ordinal can be used as a "blueprint" for a sequence of applications of the power set and union operations.
- The farther up an ordinal is in the hierarchy of ordinals, the longer the sequence, and the greater the cardinality of the end result.

Specifically, each ordinal α can be used to characterize the set $\mathfrak{B}_\alpha$:

$$\mathfrak{B}_\alpha = \begin{cases} \mathbb{N}, \text{ if } \alpha = 0 \\ \wp(\mathfrak{B}_\beta), \text{ if } \alpha = \beta' \\ \bigcup\{\mathfrak{B}_\gamma : \gamma <_o \alpha\} \text{ if } \alpha \text{ is a limit ordinal (other than 0)} \end{cases}$$

3 Later Ordinals, Bigger Cardinalities

- By Cantor's Theorem: if $\alpha <_o \beta$, then $|\mathfrak{B}_\alpha| < |\mathfrak{B}_\beta|$.
- For instance:

$$\omega <_o (\omega \times \omega) <_o \omega^\omega <_o {}^{\omega}\omega. \text{ So: } |\mathfrak{B}_\omega| < |\mathfrak{B}_{\omega\times\omega}| < |\mathfrak{B}_{\omega^\omega}| < |\mathfrak{B}_{{}^{\omega}\omega}|.$$

4 Initial Ordinals

- **Initial ordinal**: an ordinal that precedes all other ordinals of the same cardinality.
- An initial ordinal κ can be used as proxy for its own cardinality: $\kappa = |\kappa|$.

5 The Beth Hierarchy

- $\beth_\alpha$ (read "beth-alpha") is the initial ordinal of cardinality $|\mathfrak{B}_\alpha|$.
- So: $\beth_\alpha = |\mathfrak{B}_\alpha|$.
- $\beth_0 = |\mathbb{N}|$ and $\beth_{0'} = |\wp(\mathbb{N})|$ (so $\beth_{0'}$ is an **uncountable** ordinal).

Since the beths are *ordinals*, they can be used to define sets bigger than anything we've considered so far. For instance:

- $\mathfrak{B}_{\beth_{0'}}$ (where $\beth_{0'} = |\wp(\mathbb{N})|$)
- $\mathfrak{B}_{\beth_{\beth_\omega}}$ (where $\beth_{\beth_\omega} = |\mathfrak{B}_{\beth_\omega}|$)

6 The Continuum Hypothesis

Continuum Hypothesis There is no set A such that $\beth_0 < |A| < \beth_1$.

Generalized CH There is no set A such that $\beth_\alpha < |A| < \beth_{\alpha+1}$.

7 The Burali-Forti Paradox

Suppose, for *reductio*, that Ω is the set of all ordinals. Then:

- Since Ω consists of every ordinal, it consists of every ordinal that's been introduced so far. But a new ordinal is just the set every ordinal that's been introduced so far. So: Ω **is an ordinal**.
- If Ω was itself an ordinal, it would be a member of itself (and therefore have itself as a predecessor). But no ordinal can be its own predecessor. So: Ω **is not an ordinal**.

So there is no set of all ordinals!

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