Problem a1

Answers may vary, but anything along the lines of, we expect this to be a great class, will be accepted.

Problem a2

a2-1)

Wind Turbine

- Boundary: Wind turbine blades, connection to the earth, connection to power grid
- Inputs: Wind velocity (speed+direction), atmospheric pressure/temperature/humidity, disturbances
- Outputs: Power, Volume/Height, Noise

Cable Stayed Bridge

- Boundary: Connection to street, connection to ground, entire surface for wind induced vibrations
- Inputs: Traffic loads and variation, external loads such as wind or water
- Outputs: Size, cost

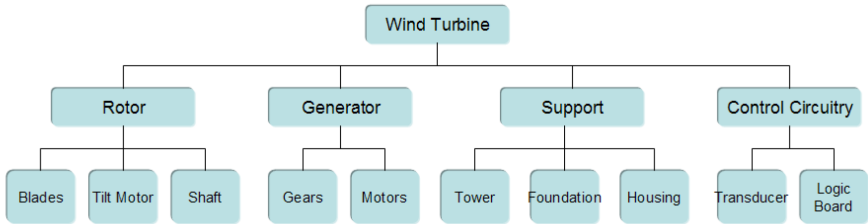
Plug-in-hybrid Electric Car

- Boundary: Tire surface to road, the plug, fuel tank nozzle
- Inputs: Number of passengers, voltage from wall, gas, driving conditions-break or gas pedal use
- Outputs: Fuel economy, driving performance, cost, pollution, noise

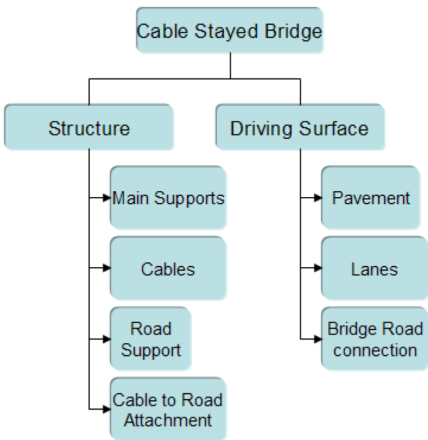
Submarine

- Boundary: Submarine structure (hull)
- Inputs: water (ballasts), communications signals (electromagnetic), sensing signals (SONAR)
- Outputs: water (ballasts), communications signals, SONAR, propelling forces, weapons

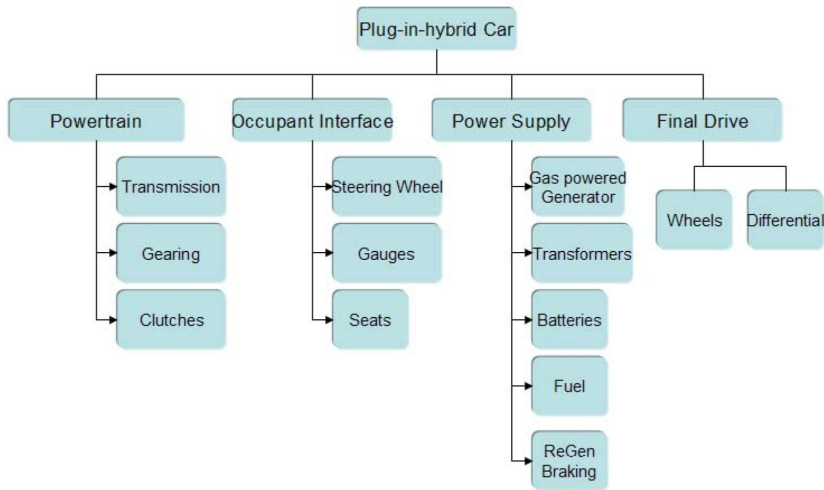
a2-2)



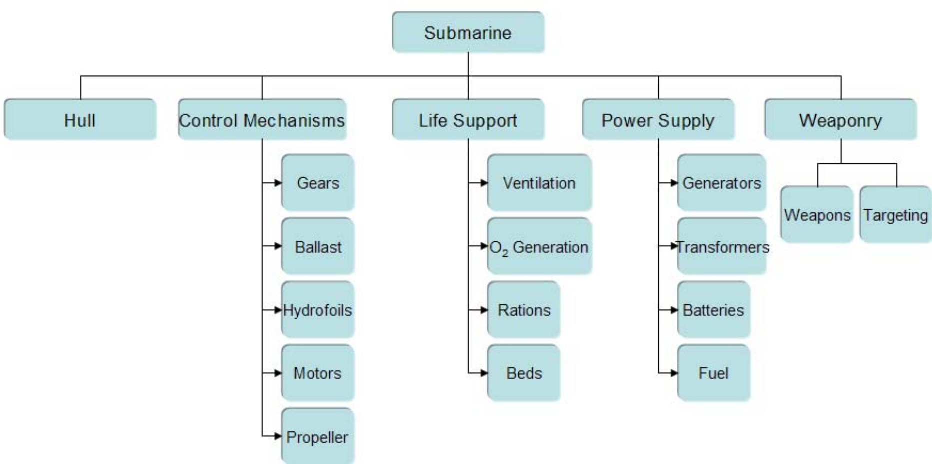
(a) Wind Turbine.



(b) Cable stayed bridge.



(c) Plug-in-hybrid electric car.



(d) Submarine.

Figure 1: System component descriptions.

a2-3)

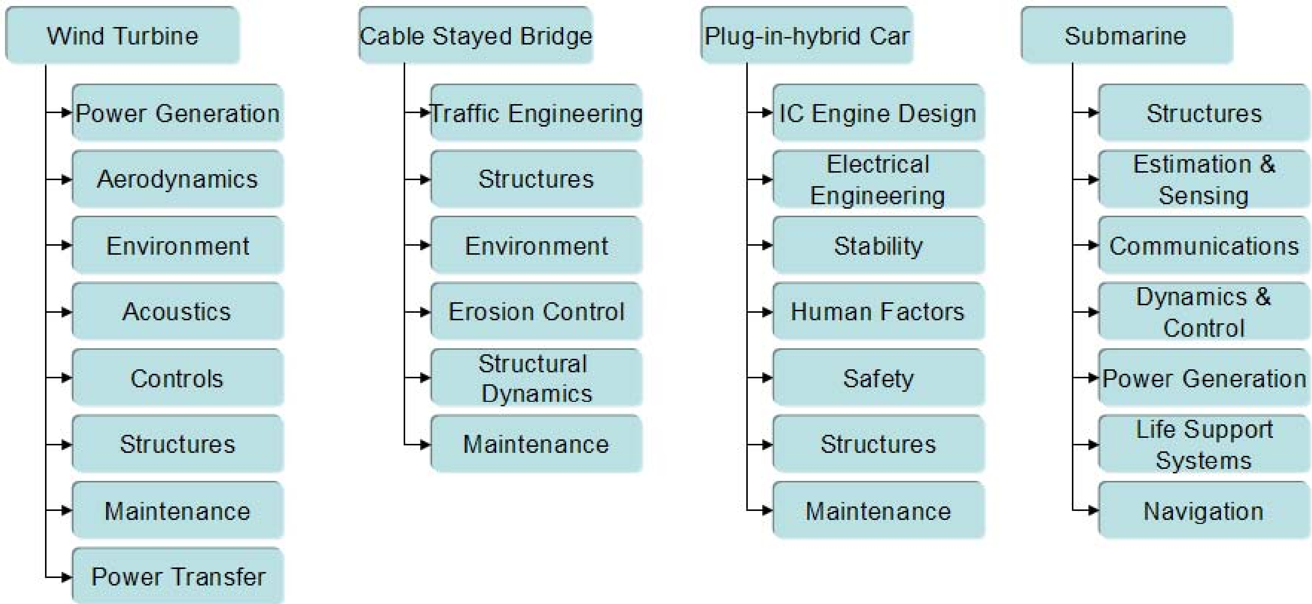


Figure 2: System decomposition by aspect.

a2-4)

(a) Sketch and interpret the following function:

$$f(\mathbf{x}) = x_1^2 + 5x_2^2 - 2x_1^3 x_2 + x_1^4 + 2x_2^4 \quad \forall \mathbf{x} \in [-5, 5] \tag{1}$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - 6x_1^2 x_2 + 4x_1^3 \\ 10x_2 - 2x_1^3 + 8x_2^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

This shows $\mathbf{x} = (0, 0)$ is the only stationary point, to check to see if this point is a minimum, maximum, or saddle point, the Hessian must be checked.

$$H(\mathbf{x}) = \nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 - 12x_1 x_2 + 12x_1^2 & -6x_1^2 \\ -6x_1^2 & 10 + 24x_2 \end{bmatrix} \tag{4}$$

$$H(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix}. \tag{5}$$

$H(0,0)$ is positive definite so the point $\mathbf{x} = (0,0)$ is a local minimizer of $f(\mathbf{x})$. From the plot, Figure 3, it is clear that $\mathbf{x} = (0,0)$ is actually the global minimizer of $f(\mathbf{x})$.

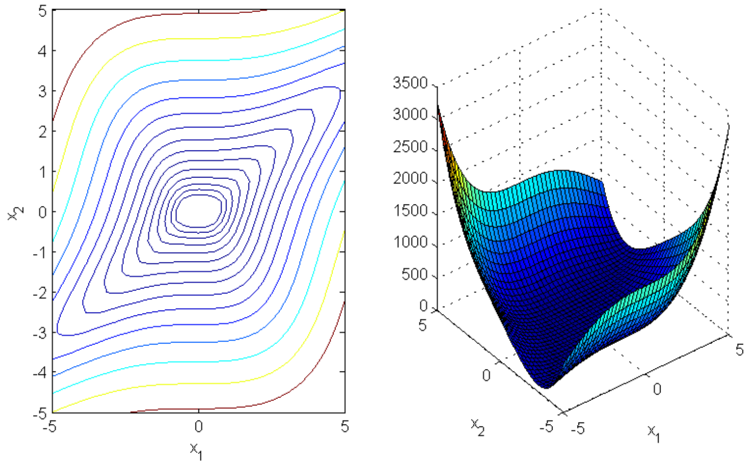


Figure 3: $f(\mathbf{x}) = x_1^2 + 5x_2^2 - 2x_1^3x_2 + x_1^4 + 2x_2^4 \quad \forall \mathbf{x} \in [-5, 5]$.

(b) Same as above, but add the constraint,

$$g(\mathbf{x}) = (x_1 - 3)^2 + 2x_2^2 + 3x_1x_2 - 2 \leq 0 \tag{6}$$

$g(0,0) = 7$, so the constraint is violated at the local minimum $\mathbf{x} = (0,0)$. This means the constraint will be active at the constrained minimum, $g(\mathbf{x}^0) = 0$. The minimum is now at, $\mathbf{x}^* = [1.2608, -0.3278]^T$. See Figure 4.

Problem a3

a3-1)

Given the function:

$$f(\mathbf{x}) = x_1^4 - x_1^2x_2 + x_2^2 + \frac{1}{2}x_1^2 \tag{7}$$

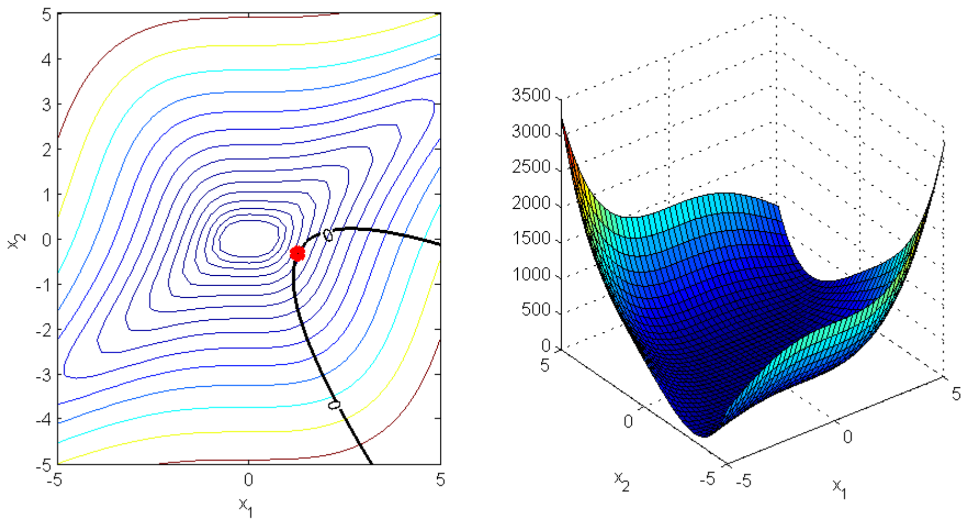


Figure 4: $f(\mathbf{x}) = x_1^2 + 5x_2^2 - 2x_1^3 x_2 + x_1^4 + 2x_2^4 \quad \forall \mathbf{x} \in [-5, 5]$ and constraint $g(\mathbf{x}) = (x_1 - 3)^2 + 2x_2^2 + 3x_1 x_2 - 2 \leq 0$.

Compute the gradient and Hessian of $f(\mathbf{x})$. Show that $\mathbf{x}^* = (0, 0)$ is the only local minimize of this function and that the Hessian matrix at that point is positive definite.

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1^3 - 2x_1 x_2 + x_1 \\ -x_1^2 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} \pm\frac{\sqrt{3}}{3} i \\ -\frac{1}{6} \end{bmatrix} \tag{9}$$

This shows $\mathbf{x} = (0, 0)$ is the only stationary point. To show that it is a local minimum, we must check the Hessian, $H(0, 0)$:

$$H(\mathbf{x}) = \nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 12x_1^2 - 2x_2 + 1 & -2x_1 \\ -2x_1 & 2 \end{bmatrix} \tag{10}$$

$$H(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \tag{11}$$

The Hessian is positive definite, so $\mathbf{x}^* = (0, 0)$ is a local minimizer, and as $\mathbf{x}^* = (0, 0)$ is the only stationary point it is accordingly the only local minimizer.

a3-2)

Make a contour plot of the objective value, $f(\mathbf{x})$, versus the design variables x_1, x_2 and verify the local minimum graphically.

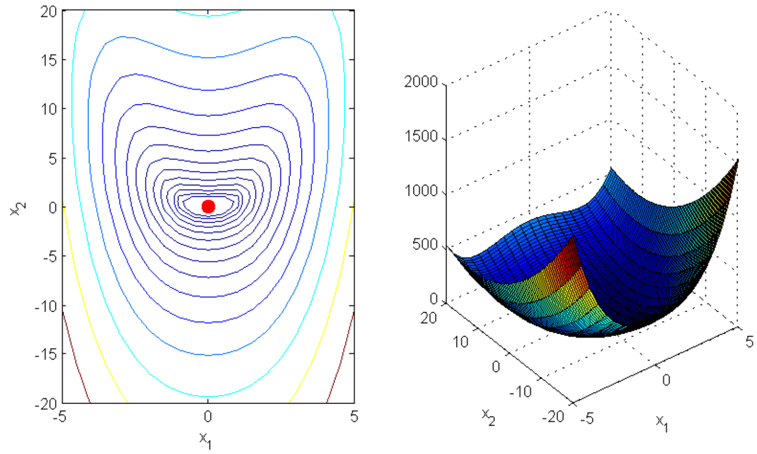


Figure 5: $f(\mathbf{x}) = x_1^4 - x_1^2 x_2 + x_2^2 + \frac{1}{2}x_1^2 \quad \forall\forall x_1 \in [-5,5], x_2 \in [-20,20]$.

a3-3)

Show that the function,

$$f(\mathbf{x}) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2, \tag{12}$$

has only one stationary point, and that it is neither a maximum or minimum, but a saddle point.

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{14}$$

This shows there is only one stationary point for $f(\mathbf{x})$ and it's at $\mathbf{x} = (1,1)$. To classify it as a maximum, minimum, or saddle point, we need to check the Hessian,

$$H(\mathbf{x}) = \nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \tag{15}$$

The eigenvalues of the Hessian are $\lambda_1 = -0.5311$ and $\lambda_2 = 7.5311$, and one is positive, and one is negative. Accordingly, $H(1,1)$ is indefinite and $\mathbf{x} = (1,1)$ is a saddle point. See Figure 6

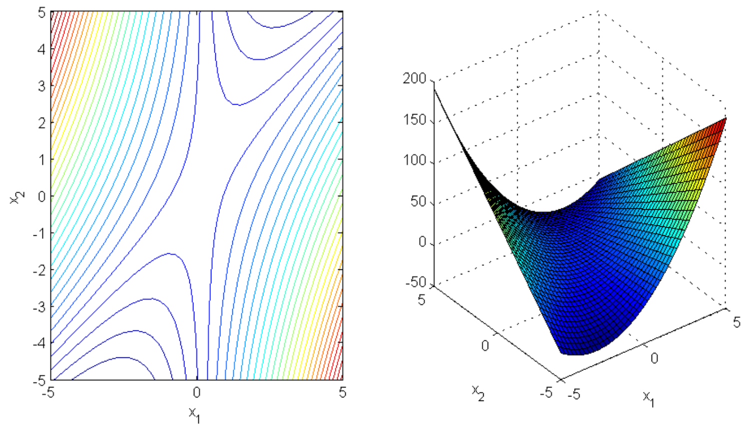


Figure 6: $f(\mathbf{x}) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2 \quad \forall \mathbf{x} \in [-5, 5]$.

a3-4)

How many stationary points does the function,

$$f(\mathbf{x}) = \frac{1}{3}x_1^3 + x_1x_2 + \frac{1}{2}x_2^2 + 2x_2 - 5, \tag{16}$$

have? Classify all of the stationary points as either maximum, minimum, or saddle points.

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2 \\ x_1 + x_2 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ -1 \end{bmatrix} \tag{18}$$

To classify the two stationary points, the Hessian must be checked:

$$H(\mathbf{x}) = \nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2x_1 & 1 \\ 1 & 1 \end{bmatrix} \tag{19}$$

$$H(2,-4) = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \tag{20}$$

$$H(-1,-1) = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \tag{21}$$

$H(2,-4)$ is positive definite, so $\mathbf{x}^* = [2,-4]^T$ is a local minimum. $H(-1,-1)$ is indefinite, so $\mathbf{x}^* = [-1,-1]^T$ is a saddle point. See Figure 7.

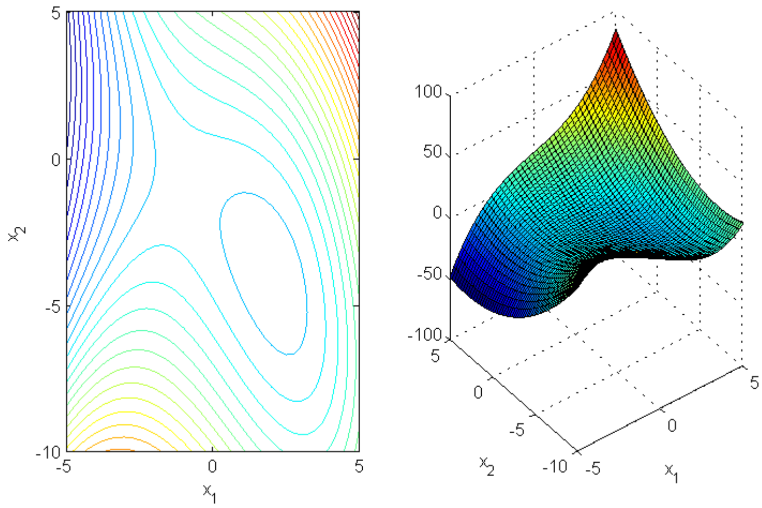


Figure 7: $f(\mathbf{x}) = \frac{1}{3}x_1^3 + x_1x_2 + \frac{1}{2}x_2^2 + 2x_2 - 5 \quad \forall x_1 \in [-5,5], x_2 \in [-10,5]$.