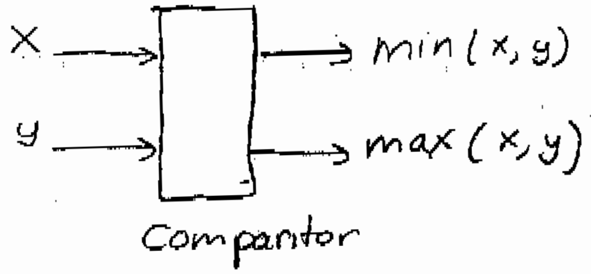
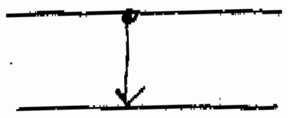
Wed 3/17
Michael Bender Lecturing

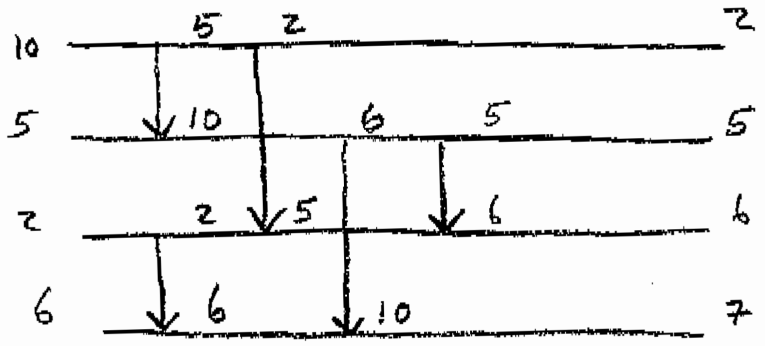
Comparison Networks



Notation:



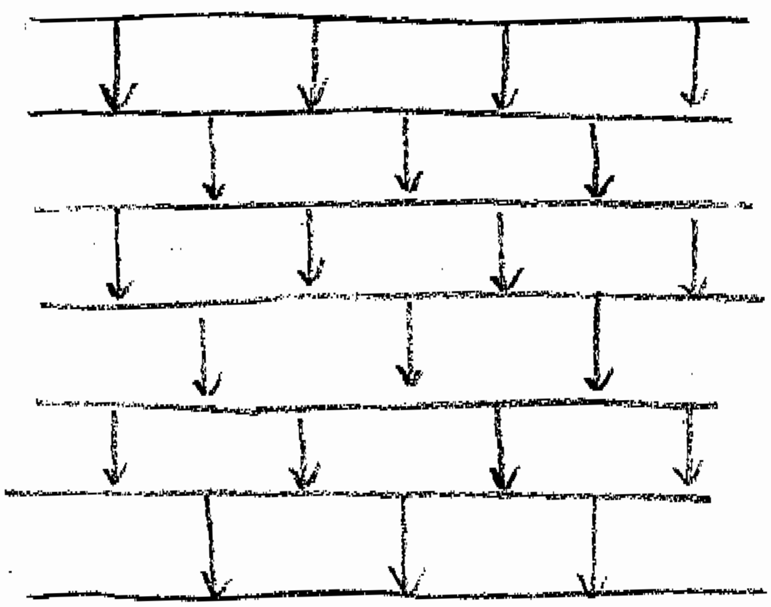
Sorting Network [developed in 50s]



Sorted outputs
<<why does it sort?>>

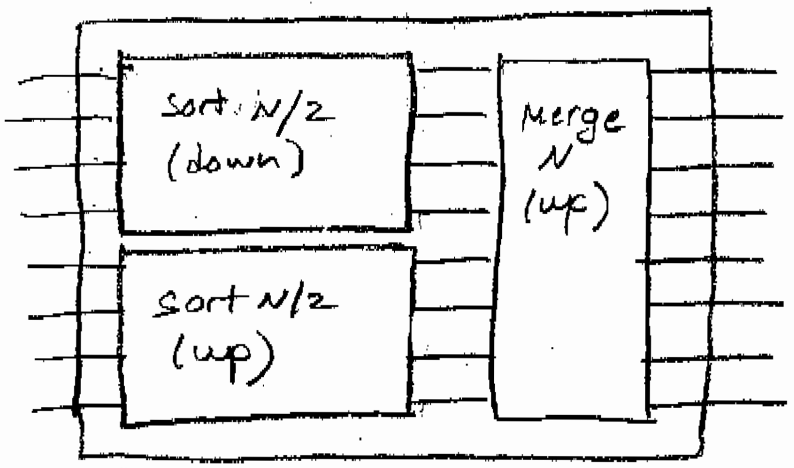
Running time = depth = longest path of comparitors (=3)

Odd-Even Transposition Sort



Depth = N
<<how low can you go?>>

step3 — Sorting network = mergesort [Batcher]



Depth $D(N) = D(N/2) + \lg N$ ← merge

$= \Theta(\lg^2 N)$

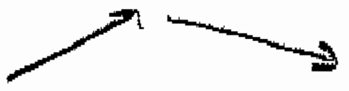
Size $S(N) = 2S(N/2) + \Theta(N \lg N)$

$= \Theta(N \lg^2 N)$

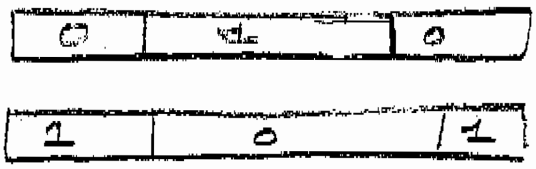
Example:

Bitonic Sorting Network (Batcher)

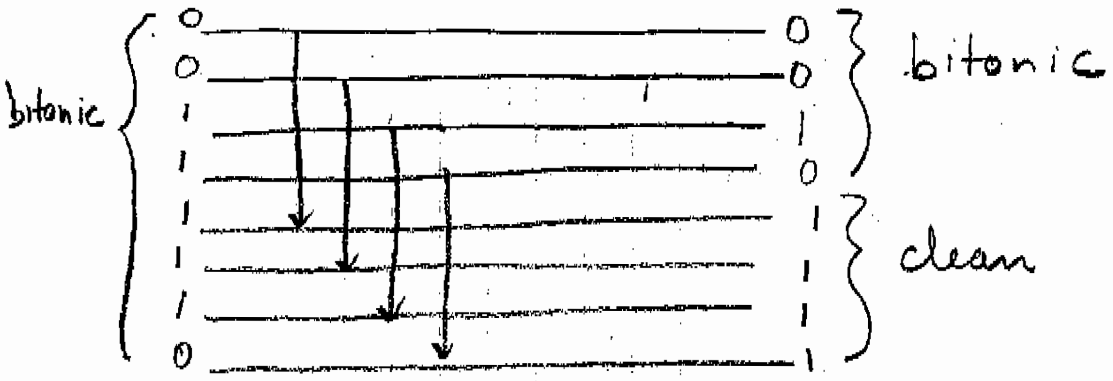
Step 1 Sort "bitonic" sequence

def: A bitonic sequence: 

or cyclic rotation

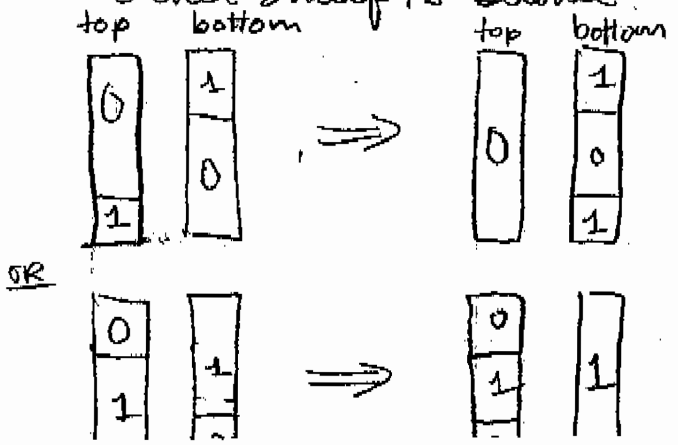
⟹ 0-1 bitonic sequence: 

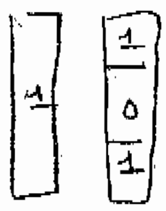
key subnetwork: half cleaner



claim: half output is clean (& bitonic)

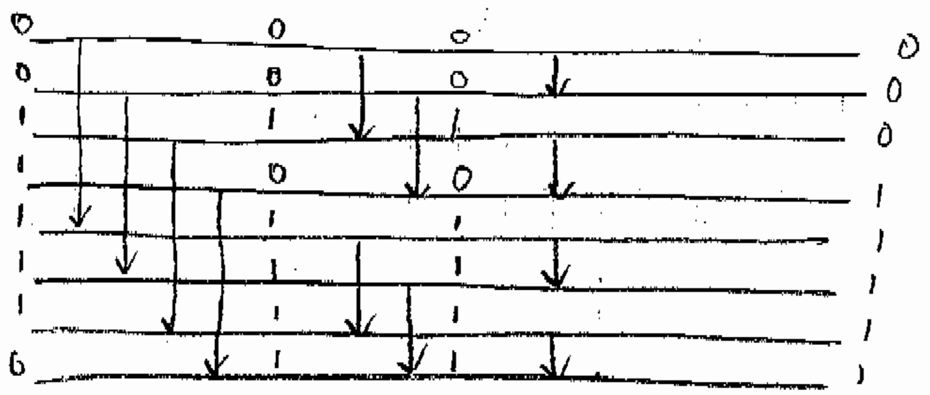
other half is bitonic

Pf: 

<< other case: 

>>

Sort bitonic sequence



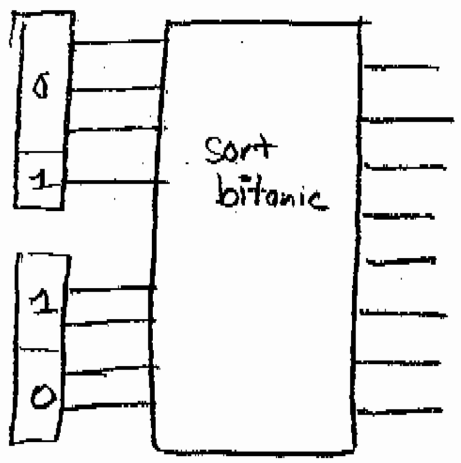
Depth: $D(N) = D(N/2) + 1$

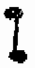
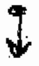
$= \lg N$

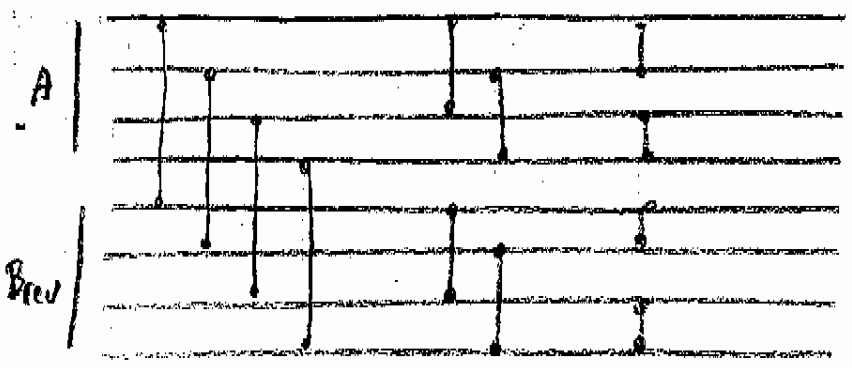
Size: $S(N) = 2S(N/2) + N/2$

$= \Theta(N \lg N)$.

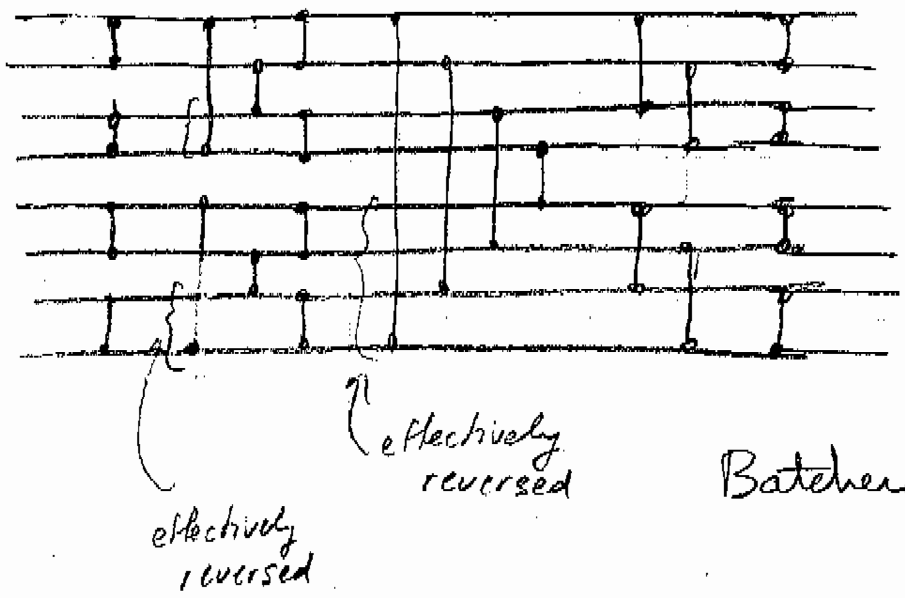
Step 2: Construct merging network, one sorted up, other down.



Frequently drawn where  means 



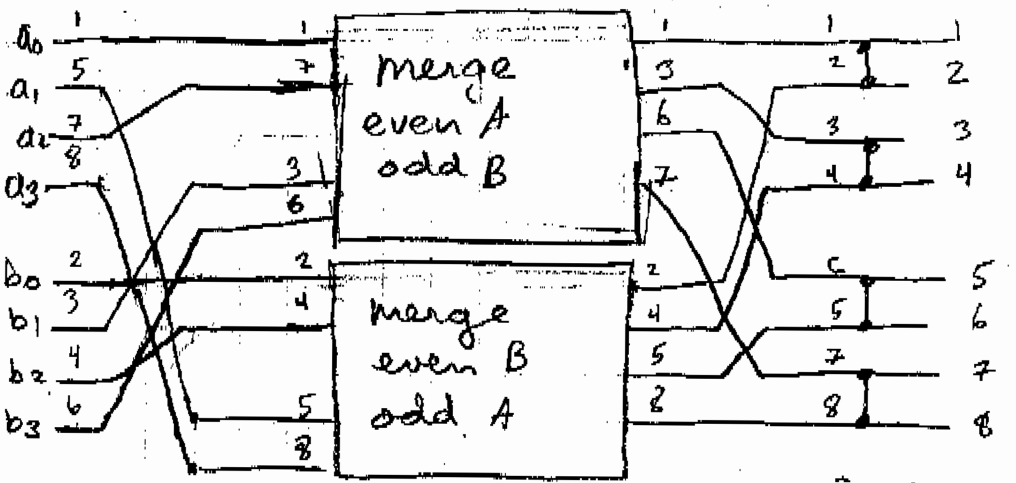
Merging Circuit



Batchers circuit

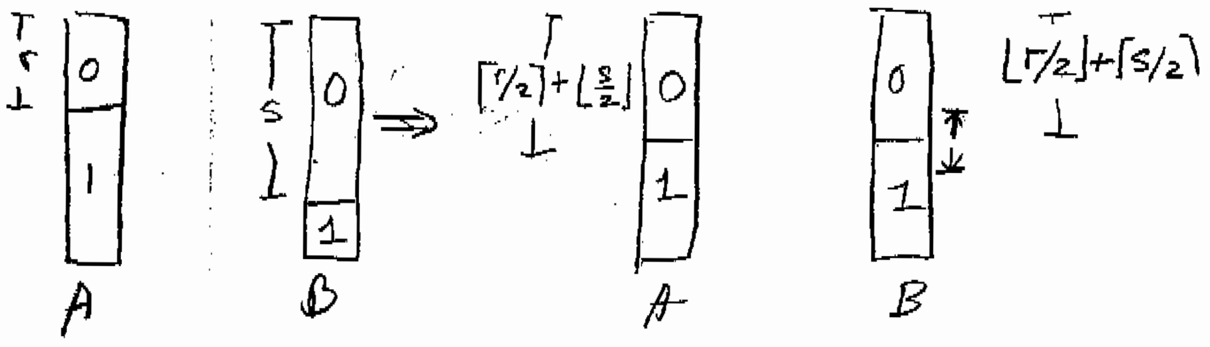
Batcher's Odd Even Merge Sort

step 1: build merger of $A = a_0 \dots a_{N/2}$ $B = b_0 \dots b_{N-1}$



Interleave elements

Proof: 0-1 Lemma.



r
0
1
A
s
0
1
B
$\lceil r/2 \rceil + \lfloor s/2 \rfloor$
0
1
A
0
1
B
$\lfloor r/2 \rfloor + \lceil s/2 \rceil$

$\Rightarrow$ #0's in each list differs by 1.

Merge $M(N) = M(N/2) + 1$
$= \Theta(N \lg N)$

Sort: $S(N) = S(N/2) + M(N)$

$= \Theta(N \lg^2 N)$.

Longstanding open question:

Does there exist sorting network with depth $O(\lg N)$?

1983: yes! AKS sorting network (Ajtai, Komlós, Szemeré

Depth: N

comparators: $O(N \lg N)$

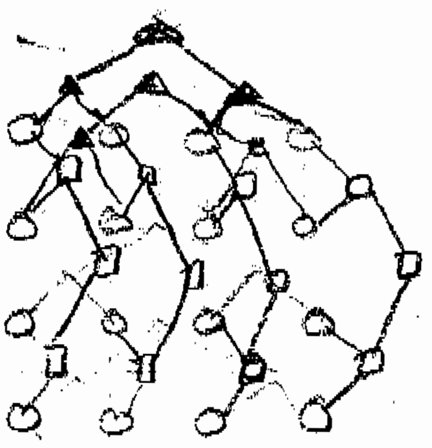
Unfortunately VERY large constants: many thousands!

Sorting on Mesh of Trees.

<u>Def.</u> 2 dimensional mesh of trees (MOT) $M_{2,N}$.

N×N Grid → · remove grid edges

· add tree above every row & column



Nodes: $N\ (2N-1) + N\ (N-1) = 3N^2 - 2N$

diameter: $4 \lg N$

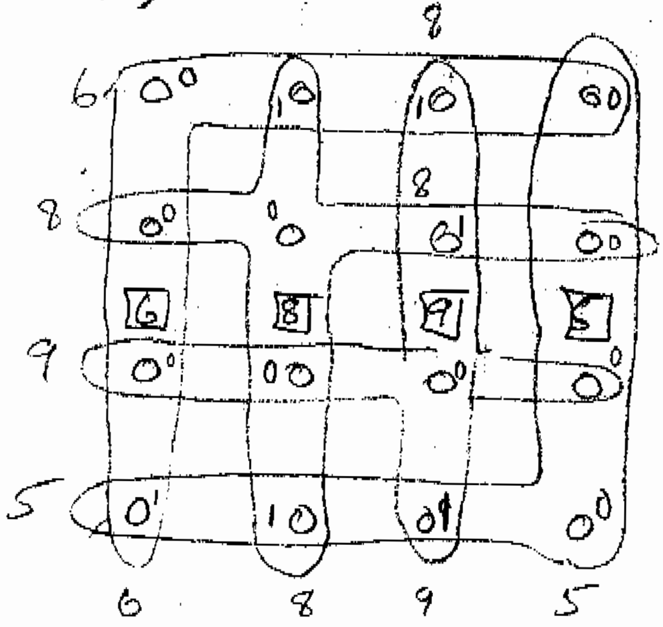
bisection width: N

recursive decomposition: remove all roots

$\Rightarrow$ 4 separate $M_{2,N/2}$.

Sort: N^2 elmts: $\Omega(N)$ time (bisection LB)

Sort N elmts: $\Theta(\lg N)$ time
($w_1 \ldots w_N$)



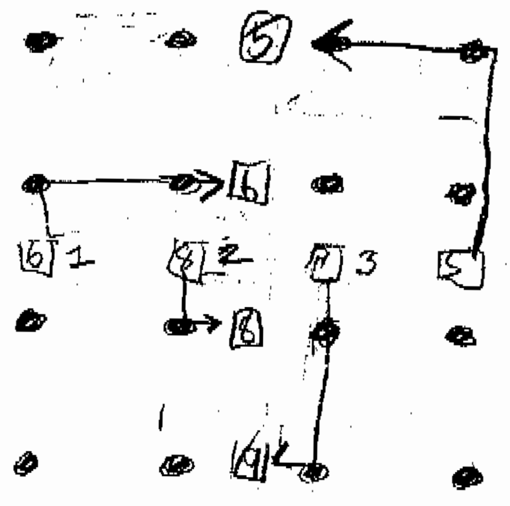
(1) pass w_i along ith row & column

(2) In node p_{ij} (row i, column j) store $\begin{cases} 1, & w_i \le w_j \\ 0, & w_j > w_i \end{cases}$

(3) count #1's in jth tree
⇒ rank of w_j in sorted order

(4) if $\text{rank}(w_j) = r$
⇒ send w_j to rth row tree.

⇒



<<Wow. Routing! >>

Note: $\Theta(\overset{2k+5\lg N}{k+\lg N})$ bitsteps for k bit #s.

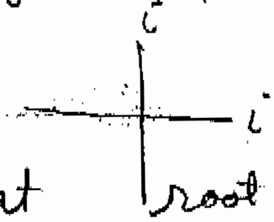
Send MSB first.

Sort N^2 elmts: $\Omega(N)$ time (bisection LB)

Sort N elmts: $w_1, w_2, \ldots, w_N$ $\Theta(\lg N)$ time

<u>Idea:</u> brute force. Do all comparisons.

Given N k-bit #s, following bit-step alg sorts in $2k + 5\lg N$ ste

(1) $\forall i$, pass w_i along  ith column & row (MSB first) Store at root of each column tree.

(2) For each leaf, bitwise compare w_i & w_j.

(Break ties with index i, j)

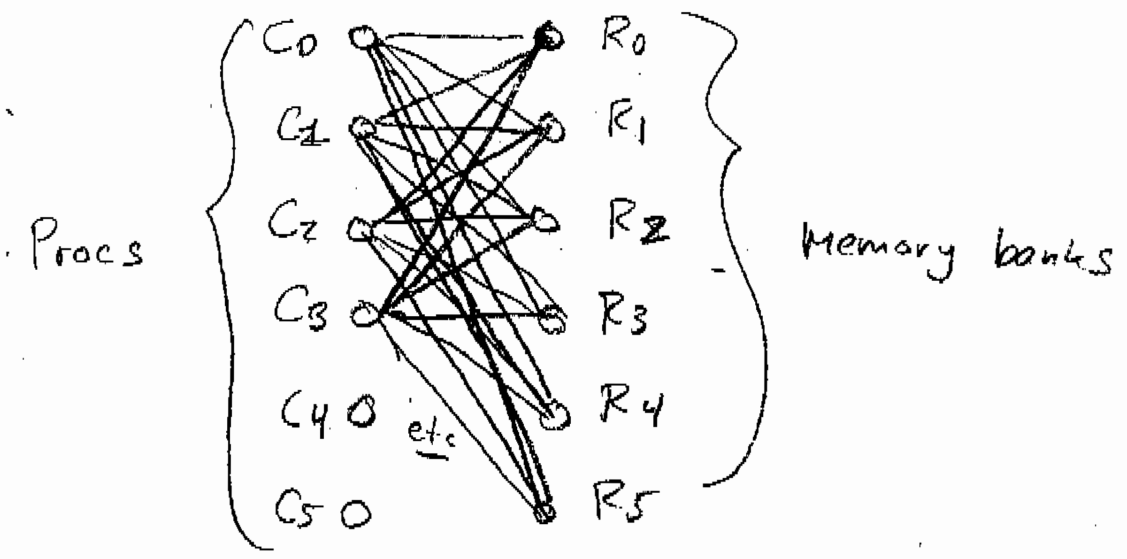
Leaf p_{ij} stores $\begin{cases} 1 & \text{if } w_i < w_j \\ 0 & w_i \geq w_j \end{cases}$

(3) $\forall j$, count # 1's in leaves of jth column tree

$\Rightarrow$ rank of w_j in sorted order.

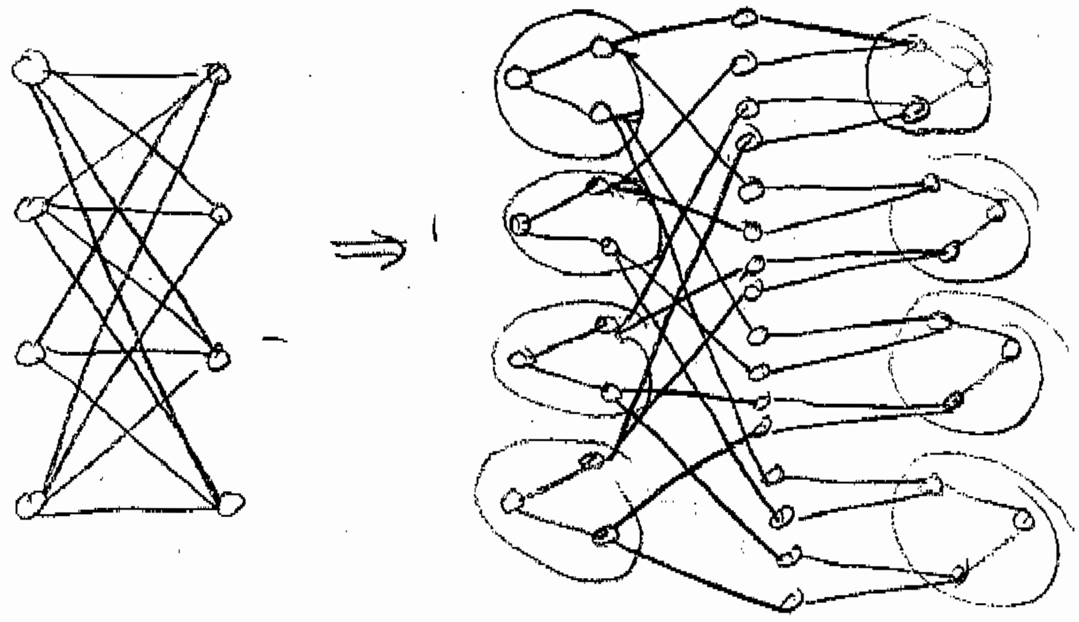
(4) If rank$(w_j) = r$, <u>send</u> w_j to root of rth row tree

Simulating Bipartite Graph / Ideal Computer on MOT



For large N, K_{NN} not realistically implementable.

$\Rightarrow$ ~~Simulate~~ K_{NN} by M_{2N} with $2\lg N$ delay



<u>The catch:</u> quadratic blowup in space/hardware