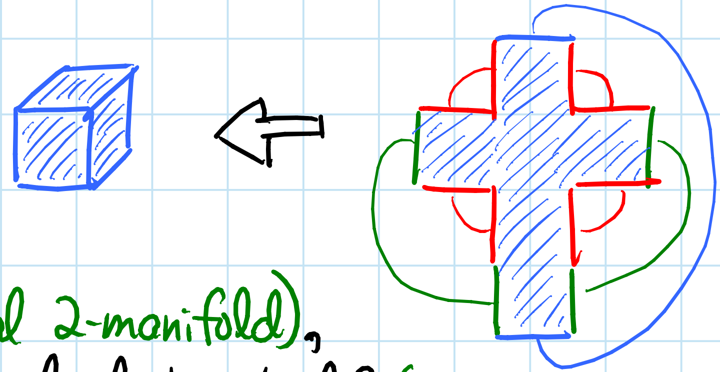
| 6.849 | Lecture 15 | Nov. 1, 2010 |
|---|---|---|



**Folding polyhedra:**

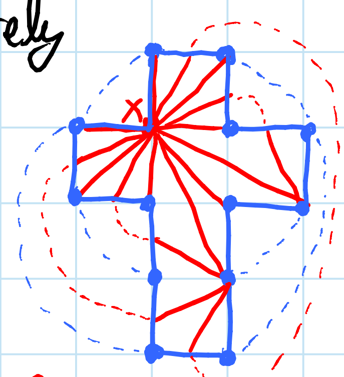
**Decision problem:**
given a polygon
(or connected metric polygonal 2-manifold),
can its boundary be glued to itself (in pairs of intervals)
such that resulting surface can be folded
into exactly a convex polyhedron?
↳ no multiple layers like origami

**Enumeration problem:** list all gluings & foldings

**Combinatorial problem:** how many can there be?

**Why convex polyhedra?** always possible to
fold into a (nonconvex) polyhedron
provided orientable or some unglued boundary
[Burago & Zalgaller 1960, 1996; O'Rourke 2010]

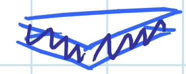
**Alexandrov gluing:** polygon + gluing induce a metric
by shortest-path lengths between all pairs of points
- metric is polyhedral: all but finitely many points have zero curvature
- metric is convex if all points have zero or positive curvature
- metric is topological sphere if gluing noncrossing




shortest paths from x to all vxs.

# Alexandrov's Theorem: [1941; English book 2005]

every convex polyhedral metric, topologically a sphere, is realized by a unique convex polyhedron
(possibly degenerating to doubly covered flat polygon)

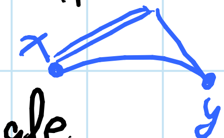


## Proof sketch: [Lecture 14]

Uniqueness: draw all shortest paths between pairs of vxs.
- includes all edges of any polyhedral realization

$\Rightarrow$ faces between mesh of paths are rigid
- Cauchy's Rigidity Theorem $\Rightarrow$ unique convex realiz.

Existence: induct on $n = \#$vertices
- base case: $n \le 4$ (double triangle or tetrahedron)
- total curvature of all vertices $= 720° = 4\pi$
  [Descartes' Theorem; conseq. of Gauss-Bonnet Formula]
- $n \ge 5 \Rightarrow 2$ vertices $x, y$ have curvatures $\alpha, \beta < 180°$
- along shortest path from $x$ to $y$, paste edge of a doubly covered triangle




  $\Rightarrow$ new vertex @ triangle apex; adds material @ $x$ & $y$
- continuously vary angles of triangle at $x$ & $y$ from $\emptyset$ to $\alpha/2$ & $\beta/2$ $\Rightarrow$ $x$ & $y$ flatten

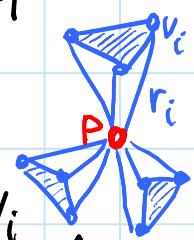
$\Rightarrow$ continuous path on manifold of metrics from original metric to metric with one less vertex
- induct on latter — lost $x$ & $y$, gain apex
- argue continuity of realizability using Implicit Function Theorem $\Rightarrow$ nonconstructive $\square$

# Constructive Alexandrov's Theorem: [Bobenko & Izmestiev 2006]

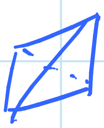
(following Blaschke & Herglotz 1937; Alexandrov 1950; Volkov 1955)

Idea: represent interior of polytope, not just boundary
- add (hypothetical) point $p$ interior to polytope
- triangulate surface with geodesics
- form solid tetrahedron on $p$ & each $\Delta$
- solve for distance $r_i$ from $p$ to vertex $v_i$

$\Rightarrow$ determines geometry of tetrahedra, hence polytope




Generalized polytope: same combinatorial structure, tetrahedra glued around $p$, but not necc. in 3D



- consider dihedral angles of edges of tetrahedra ~ view as angle of solid material
- convexity invariant: $\Sigma$ two dihedral angles incident to edge of surface triangulation $\leq 180°$
- goal: reach real polytope where $\kappa_i = 360° - \Sigma$ dihedral angles around interior edge $(p, v_i) = \emptyset$

Evolution: start at generalized polyhedron $P(\emptyset)$
- set $\kappa_i(t) = (1-t)\,\kappa_i(\emptyset) \to \emptyset$ as $t \to 1$
- differential equation to evolve $r_i$'s:

$$\frac{d\vec{r}}{dt} = \underbrace{\left(\frac{\partial \vec{\kappa}}{\partial \vec{r}}\right)^{-1}}_{\text{Jacobian - how } r_i\text{'s affect } \kappa_j\text{'s}} \cdot \vec{\varphi}(\emptyset)$$

- geodesic triangulation changes (flips) as $t \to 1$
- crucial part: Jacobian nonzero & has inverse (uses inverse function theorem!)

# Constructive Alexandrov's Theorem: (cont'd)

Starting point: need a generalized polyhedron $P(\emptyset)$
1. geodesic Delaunay triangulation of surface
2. setting all $r_i$ equal & sufficiently large yields desired convexity invariant
   - using Delaunay property

## Pseudopolynomial algorithm for Alexandrov's Theorem:

[Kane, Price, Demaine 2009]

$O(n^{456.5}\, r^{1891} / \varepsilon^{121})$ time

$\varepsilon$ → accuracy

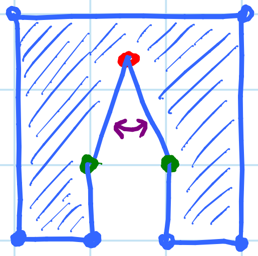
$r$ → spread = largest dist./smallest dist.

- compute geodesic Delaunay by modifying [Mitchell, Mount, Papadimitriou 1987] to handle when edges not necc. shortest paths
- make each part effective with explicit bounds:
  - how large to make initial $r_i$'s
  - Jacobian & inverse bounded away from $\emptyset$ (using Hessian instead of inverse function thm)

$\boxed{\text{OPEN}}$: polynomial time possible?
- logarithmic dependence on $r/\varepsilon$ possible: reduces to roots of $2^{O(n)}$-degree polynomial [Sabitov 1996; Fedorchuk & Pak 2005]

## Ungluable polygon: [Demaine, Demaine, Lubiw, O'Rourke 2000]

- no vertex can be glued into red reflex vertex: $< 90°$ free



$\Rightarrow$ "zip" red reflex vertex
$\Rightarrow$ green reflex vertices glued together
$\Rightarrow$ $> 360°$ of material $\square$

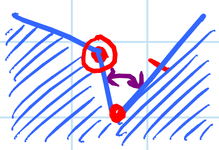
## Random polygons are ungluable:

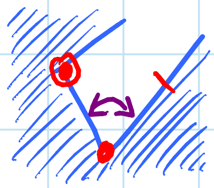
- suppose uniform distribution on angles & edge lengths

$\Rightarrow \approx n/2$ reflex vertices

- gluing in a convex vertex still leaves reflex vertex (angles don't match)
- at some point must zip a reflex vertex
- fails if nearer angle is reflex:

convex $\Rightarrow$ OK





reflex $\Rightarrow$ BAD

- happens with probability 1/2 for each reflex vertex $\square$

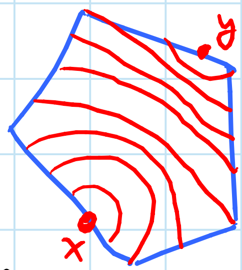
**Perimeter halving:** every convex polygon has an Alexandrov gluing

- pick any point $x$ on polygon boundary
- glue together two boundary points at distance $d$ from $x$ (measured along boundary), for all $d > \emptyset$
  - both points have $\leq 180°$ of material $\Rightarrow$ convex
- stop at diametrically opposite point $y$

$\Rightarrow$ gluing two halves (paths) of perimeter from $x$ to $y$

- $x$ & $y$ also convex (nothing glued)

$\Rightarrow$ Alexandrov □




**EXPERIMENT:** cut out convex polygon
tape together perimeter halves
see what convex polyhedron you get

**Mostly different:** uncountably many polyhedra

- vary $x$ near vertex $v_i$, say $d$ along edge $v_i v_{i+1}$
- $x$ & $v_i$ become distinct vertices of shortest-path distance $d$
- only finitely many vertex-vertex shortest paths for a particular polyhedron
- uncountably many choices for $d$

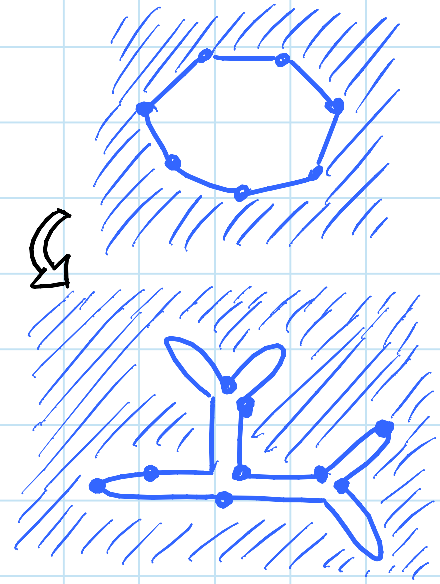
$\Rightarrow$ uncountably many polyhedra □

# Gluing tree:

- turn polygon "inside-out"
- gluing of that boundary to self forms a cycle around a tree

  gluing tree

- corresponds to cutting tree in unfolding



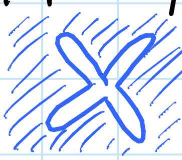
## Properties:

- each leaf is either a zipped vertex or a fold point in middle of edge ($\Rightarrow 180^\circ$)

$\Rightarrow$ at most 4 fold points ($720^\circ$ total curvature)

- if 4 fold points, then these are only leaves

  always induce curvature

  $\Rightarrow$  or 

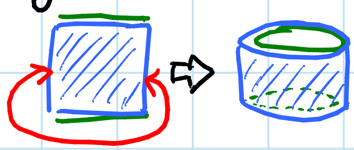
- at most one nonvertex (middle of edge) glued at $\geq 3$-way junction (else $180^\circ \cdot 2 +$ something)

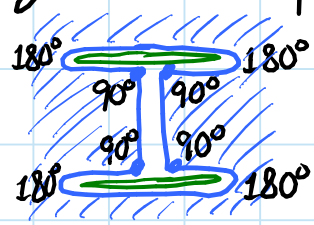
Rolling belt = path in gluing tree whose
end points are either fold pts. or convex vx. leaves
& along which always $\le 180°$ material on either side
= effectively an embedded convex polygon
$\Rightarrow$ can perimeter halve arbitrarily = "rolling the belt"
– only way to get infinite gluings

Examples:

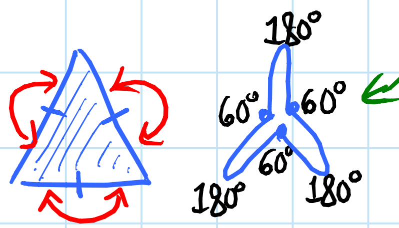
1 rolling belt: perimeter halving of convex polygon

2 rolling belts: cylinder






3 rolling belts:




belt between every pair of leaves

$\ge 4$ rolling belts: impossible [6.885 Fall 2004 PS5.3]
– must be 4 fold points
$\Rightarrow$ no curvature elsewhere
$\Rightarrow$ rolling belt from one fold point
is uniquely determined to some fold point
$\Rightarrow$ same rolling belt from latter fold point
$\Rightarrow$ $\le 2$ rolling belts