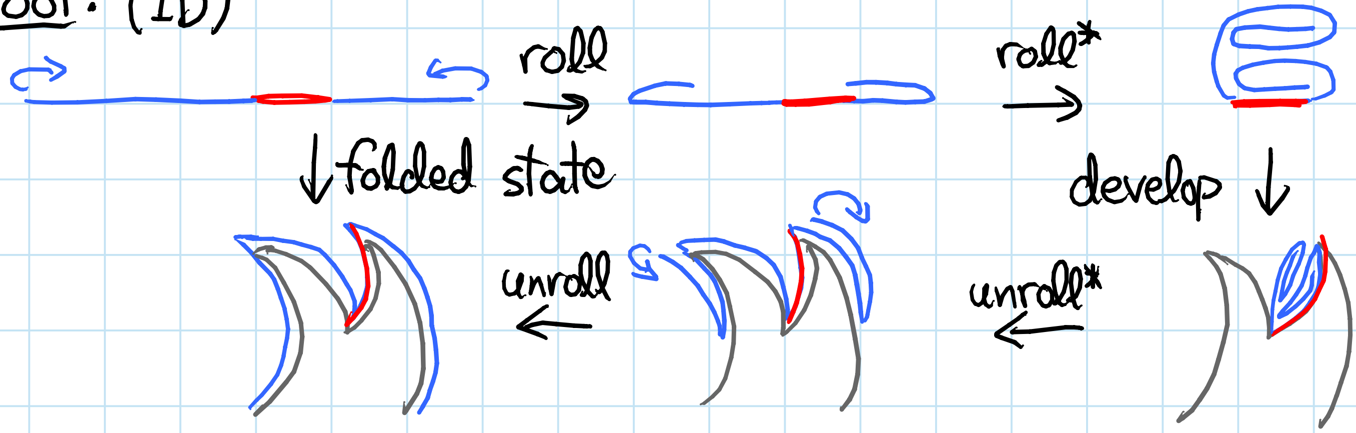
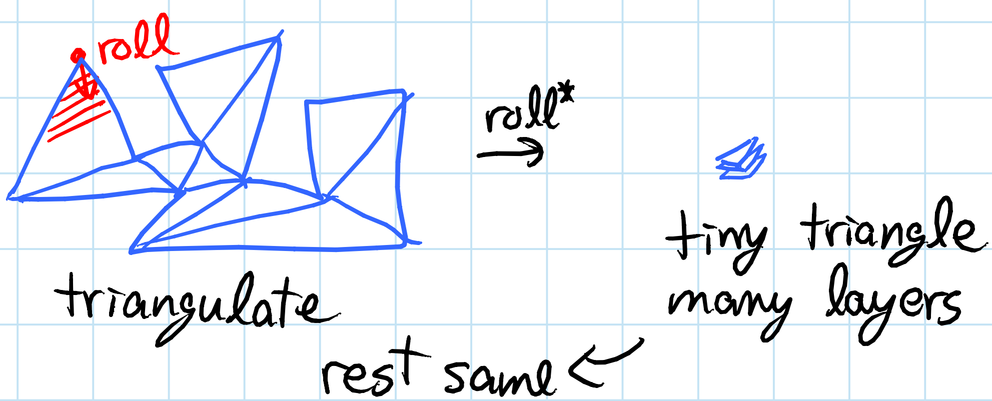
2 meanings of "folding": (origami)
- folded state = description of paper after folding
- folding motion = continuum of folded states
- we've focused on states, but in reality want motion

Equivalence: [Demaine, Devadoss, Mitchell, O'Rourke 2004]
any simple polygonal piece of paper has a folding motion into any desired folded state

Proof: (1D)



(2D)



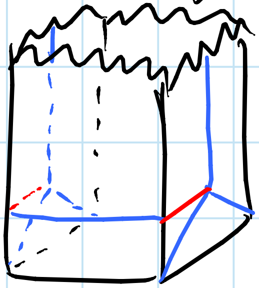
□

OPEN: what if paper has holes?
unknotted polyhedral paper? (for flattening)

OPEN: do finite number of extra creases suffice, if target folded state does not touch itself?
- above, all points become crease points

Rigid origami: what folds without extra creases?
- faces of crease pattern = rigid polygons
- creases = hinges

Example: [Balkcom, Demaine, Demaine, Ochsendorf, You 2006]
paper shopping bag doesn't fold rigidly
(for $\underbrace{h}_{\text{height}} > \underbrace{w}_{\text{width}}/2$, standard crease pattern)



- 2 folded states: open & flat
- no folding motions

Little known about rigid origami: looking for good open questions

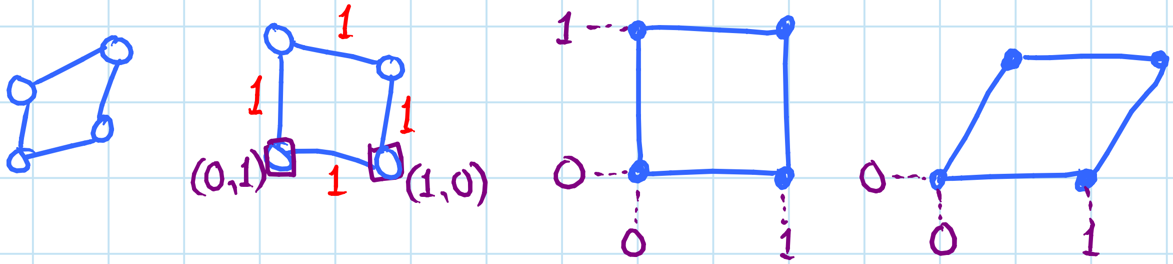
LINKAGES:

Graph = vertices (V) & edges (E) 
(connectivity/combinatorial structure)

Linkage = graph + lengths of edges ($\ell : E \to \mathbb{R}^{\geq 0}$)
(intrinsic geometry)
[+ coordinates for *pinned* vertices ($p : V' \to \mathbb{R}^d$)] where $V' \subseteq V$

Configuration of a linkage in $\mathbb{R}^d$
= coordinates for vertices ($C : V \to \mathbb{R}^d$)
satisfying constraints of linkage
$$\left(\begin{array}{l} \|C(v) - C(w)\| = \ell(v,w) \text{ for all } \{v,w\} \in E; \\ C(v) = p(v) \text{ for all } v \in V' \end{array} \right)$$
(allowing intersections for this lecture)

Example:



graph   linkage   two configurations

Motion (of a linkage in $\mathbb{R}^d$)
= continuum of configurations ($m : [0,1] \to \mathcal{C}$)

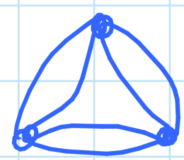
Configuration space = all configurations of a linkage

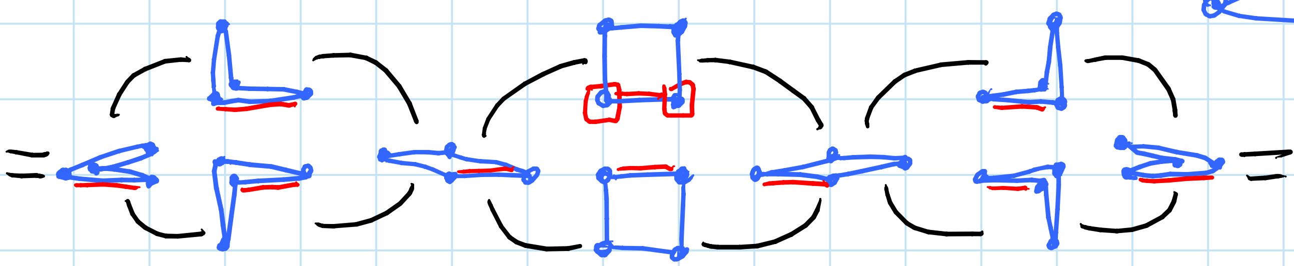
- view configuration of n-vertex linkage in $\mathbb{R}^d$ as (special) point in $\mathbb{R}^{dn}$:

$$C = (\underbrace{\dots, \dots, \dots}_{d \text{ coords for } v_1}; \underbrace{\dots, \dots, \dots}_{v_2}; \bullet\bullet\bullet; \underbrace{\dots, \dots, \dots}_{d \text{ coords. for } v_n})$$

$\Rightarrow$ configuration space = subspace of $\mathbb{R}^{dn}$

- motion = path/curve in configuration space
- square example: $n=4$, $d=2$
  - $\Rightarrow$ configuration space lives in $\mathbb{R}^8$
  - 4 dimensions fixed by pinning
  - locally one dimensional; topologically:





Degrees of freedom = local intrinsic dimension of configuration space around configuration

- intuitively: $d \cdot (\#\text{ unpinned vertices}) - (\#\text{ edges})$
  (but in reality, some edges are extraneous — see L3)

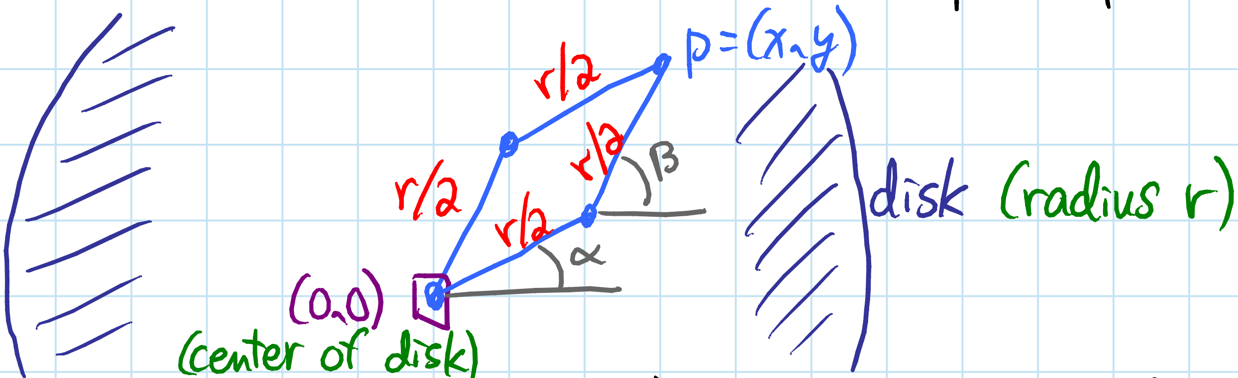
Trajectory of a vertex in a linkage = all points that vertex can reach in configurations
(= projection of configuration space onto vertex's coords)

Kempe's Universality Theorem: [Kempe 1876 had bug; Thurston; King 1999; Kapovich & Millson 2002; Abbott, Barton, Demaine 2008]

Any algebraic planar curve $\varphi(x,y) = \sum_i c_i x^{p_i} y^{q_i} = 0$, intersected with any bounded disk, (necessary) is exactly the trajectory of a vertex of some linkage.

Kempe's "proof":

- start with rhombus to constrain point p within disk:



- goal: constrain $p=(x,y)$ to satisfy $\varphi(x,y)=0$

Main trick: use trig. to effectively "take logarithm"

- $x = \frac{r}{2}\cos\alpha + \frac{r}{2}\cos\beta$
- $y = \frac{r}{2}\sin\alpha + \frac{r}{2}\sin\beta = \frac{r}{2}\cos\left(\alpha - \frac{\pi}{2}\right) + \frac{r}{2}\cos\left(\beta - \frac{\pi}{2}\right)$
- apply trig. identity

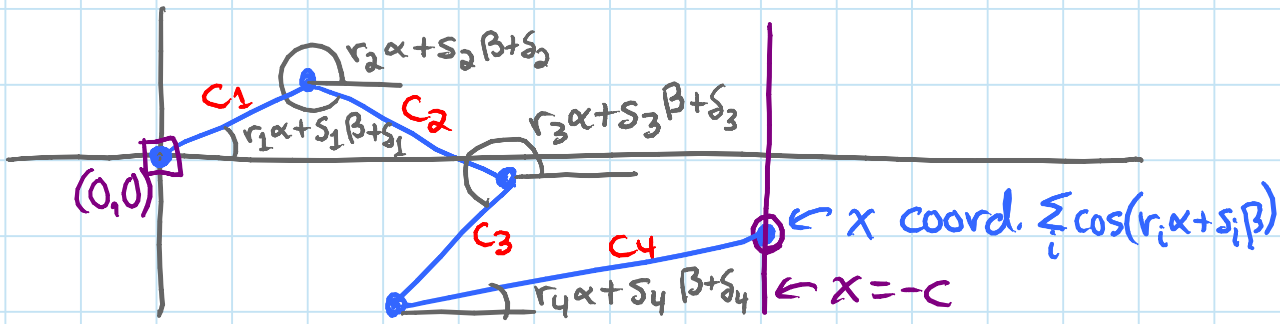
$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

to polynomial $\varphi(x,y) = \sum_i c_i x^{p_i} y^{q_i}$

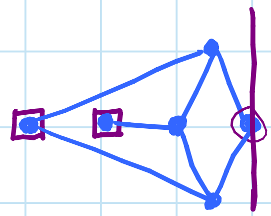
$$\Rightarrow \varphi(x,y) = \underbrace{c}_{\text{const.}} + \sum_i \underbrace{c_i}_{\text{const.}} \cos\left(\underbrace{r_i}_{\text{int.}}\alpha + \underbrace{s_i}_{\text{int.}}\beta + \underbrace{\delta_i}_{0 \text{ or } \pm\pi/2}\right)$$

Kempe's "proof": (cont'd)

- *new goal*: construct line segment of length c_i & angle $r_i\alpha + s_i\beta + \delta_i$, for each i



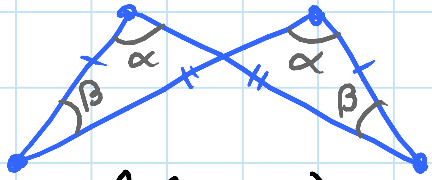
- force final vertex on line $x=-c$ via large Peaucellier linkage



- build "machine" for angle arithmetic with ops.:
  - multiply given angle by integer
  - add two given angles
  - copy an angle from one place to another

Kempe's gadgets:

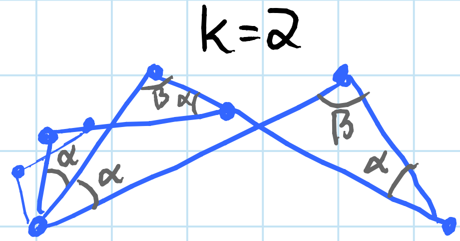
Contraparallelogram:



- opposite sides equal
  & self-crossing (not parallelogram)

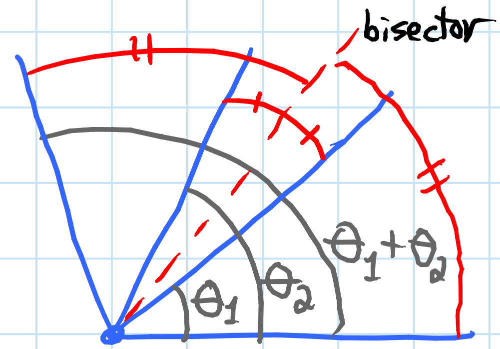
$\Rightarrow$ opposite angles equal; α determines β

Multiplicator:



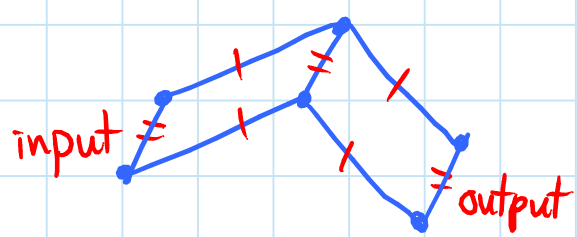
- k similar contraparallelograms
  sharing their β's $\Rightarrow$ equal α's
- can be more efficient —
  $O(\lg k)$ edges — by repeated
  doubling, but this will not affect final complexity

Additor:



- use 2× multiplicators to
  - bisect angle
    between segments
  - reflect x axis through bisector

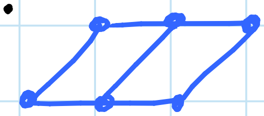
Translator: two parallelograms

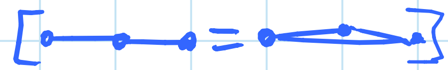


- opposite edges parallel
  & same length
- make adjacent edges long (& same) for reach
- could use big rhombus — but this construction
  allows arbitrary length of input (or output) edge

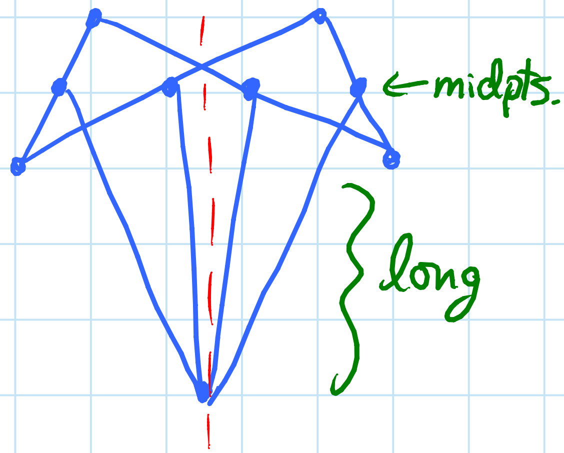
Bug: [Kapovich & Millson]

- parallelograms can flip to contraparallelograms & vice versa via degenerate (flat) configuration
  ⇒ Kempe proved weaker result:
  trajectory includes desired poly. curve & <u>more</u>
- fix for parallelogram:





- different, messier construction for complex polynomials
- fix for contraparallelogram: [Abbott & Barton 2004]



Application:

Sign name via Weierstrass approximation theorem:
any continuous function $f: [a,b] \to \mathbb{R}$ has an ε-approximate polynomial p — $|p(x) - f(x)| \le \varepsilon$ for all $x \in [a,b]$ — for any $\varepsilon > 0$
(apply to each coordinate of curve)

Generalizations/strengthenings:

- curves/surfaces in d dimensions
- $\Theta(n^d)$ bars is optimal for degree n
- any compact semialgebraic set (d-dim.) (bounded system of polynomial $\le$ inequalities) as vertex trajectory
- coNP-hard to test rigidity

[Abbott, Barton, Demaine 2008]

- configuration space = union of finitely many analytically isomorphic copies of any desired algebraic set (any # dim.)
  - mapping & inverse have local power-series expansion

[Kapovich & Millson]

$\boxed{\text{OPEN}}$: what if edges are forbidden from crossing? [Shimamoto 2004]

$\boxed{\text{PROJECT}}$: implement Kempe applet

$\boxed{\text{PROJECT}}$: sculpture based on Kempe linkage/gadgets

$\boxed{\text{PROJECT}}$: design linkages for letters of alphabet (e.g. letter C: http://www.jimloy.com/cindy/cindy.htm)

Application: constructing algebraic numbers in origami via alignments [GFALOP 19.5; cf. Alperin & Lang 2006]

6.849 Geometric Folding Algorithms: Linkages, Origami, Polyhedra
Fall 2012