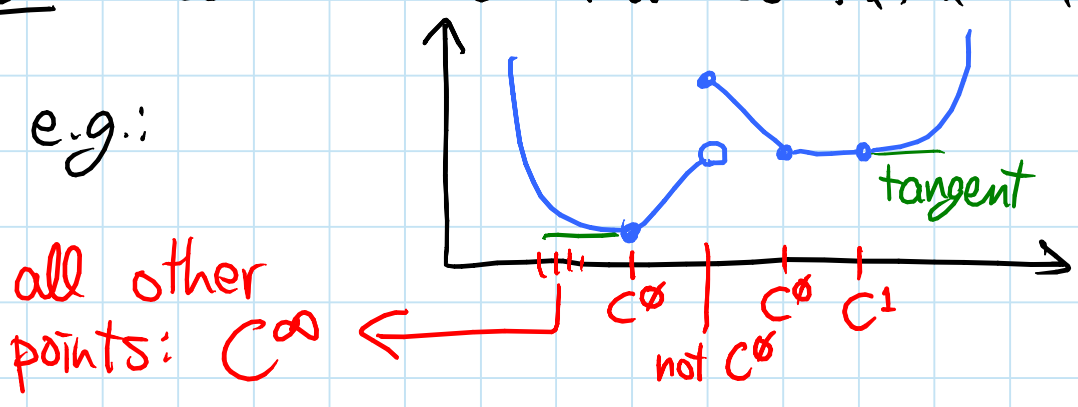
| 6.849 | Class 9 | Oct. 4, 2012 |
|---|---|---|

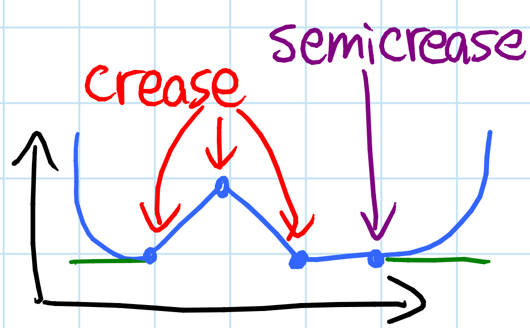
- Triangulated hypars:
  - OPEN: does alternating triangulation work for arbitrary n?
  - OPEN: do triangulations mixing alternation/not do medium well?
  - OPEN: what about regular k-gons?
    - issue: how to get started?

- Smoothness:
  - $C^\emptyset$ = continuous function f → no gaps (undefined) & no jumps
  - $C^1$ = continuous derivative $f'$
  - $C^2$ = continuous second deriv. $f''$
  - $C^\infty$ = continuous derivatives $f, f', f'', \dots$

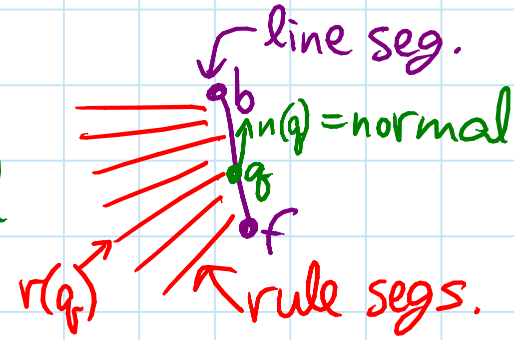
e.g.:




  - well-behaved folding = $C^\emptyset$ & piecewise-$C^2$



- **Polygonal ⇒ flat proof:**
  - $n(q) \perp bf$ } definition
  - $n(q) \perp r(q)$ } of normal




$\Rightarrow n'(q) \perp bf$

$\& \; n'(q) \perp r(q)$

... because $n(\underbrace{q+\varepsilon}_{\text{move } q \text{ along } bf}) \approx n(q) + \varepsilon \cdot n'(q)$ [Taylor series]

$$\Rightarrow 0 = n(q+\varepsilon)\cdot bf \approx \underbrace{n(q)\cdot bf}_{=0} + \underbrace{\varepsilon \cdot n'(q)\cdot bf}_{=0} \quad (\Rightarrow)$$

$$\& \; 0 = n(q+\varepsilon)\cdot r(q+\varepsilon)$$
$$\approx \underbrace{n(q)\cdot r(q)}_{=0} + \varepsilon[\underbrace{n(q)\cdot r'(q)}_{\substack{=0 \\ \text{by torsal}}} + \underbrace{n'(q)\cdot r(q)}_{=0}] + \varepsilon^2 \ldots \quad (\Rightarrow)$$

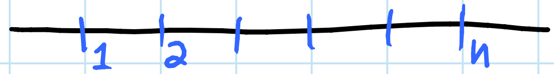
⋮

- **Mathematical vs. real paper**

  what does it all mean?

  - real paper might stretch/shear slightly
  - real paper might add many small creases (too slight to see)

$\boxed{\text{PROJECT}}$: investigate!

- Pleat folding algorithms:
  [Cardinal, Demaine, Demaine, Imahori, Ito, Kiyomi, Langerman, Uehara, Uno – G&C 2011]
  - model: 1D paper 

    n uniformly spaced creases
    some-layers simple folds
    unfold previous fold (for free)
  - look at resulting string of M/V/unfolded
  - $\approx \lg n$ folds makes "dragon curve":
    [Heighway, Banks, Harter $\approx$ 1967]

    m M v M m V m M v M v V m V v ...

    (& $\approx n$ variations)

  - MM⋯M can be folded by $\frac{3}{2} \lg^2 n$ folds
    1. fold in half until 3 creases left [xxx]
    2. fold them as M's; unfold all [MMM]
       → ? MMM ? VVV ? MMM ? VVV ? ...
    3. fold VVV's on top of each other by rep. folding middle M of middle MMM
       → [M ? VVV ? M] → [MMMMMMM]
    4. fold 5 middle creases as M; unfold all
       → VM ? $\underbrace{\text{MMMMMMM ? MVVVVVM}}$ ? etc.
    5. etc. $O(\lg n)$ rounds (doubling M reps.)
       - $O(\lg n)$ folds/round $\square$

$\Rightarrow$ MVMV$\cdots$ can be folded by $3 \lg^2 \frac{n}{2}$ folds

- MM$\cdots$M requires $\approx \frac{1}{4} \lg^2 n / \lg \lg n$ folds
  - let $f(n) = \#$ folds for $M^n \le \frac{3}{2} \lg^2 n$
  - $\Rightarrow$ fold $n/f(n)$ creases at once on average
  - $\Rightarrow$ <u>some</u> fold does $\ge n/f(n)$ creases at once
  - half of these creases become Valleys
  - need $\approx \lg n/f(n)$ folds to set this up
  - need $\approx \lg \frac{1}{2} n/f(n)$ folds to set up Mountain folding these valleys
  - etc.
  - $\Rightarrow$ need $f(n) \gtrsim \log_{\underbrace{f(n)}_{\le \frac{3}{2}\lg^2 n}} n \cdot \lg \frac{n}{\underbrace{f(n)}_{\le \frac{3}{2}\lg^2 n}} = \Omega\left(\frac{\lg^2 n}{\lg\lg n}\right)$ □

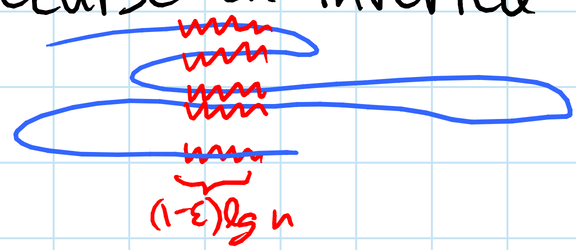
$\boxed{\text{OPEN}}$: $\Theta\left(\frac{\lg^2 n}{\lg\lg n}\right)$ or $\Theta(\lg^2 n)$ optimal?

- MVMV$\cdots$ requires $\approx \frac{1}{4} \lg^2 n / \lg \lg n$ folds too

- most M/V strings require $\approx n/\lg n$ folds
  - k folds $\Rightarrow \le (2n)^k$ choices
    M/V where
  - another $\approx 4^k$ choices for how to place $\le k$ unfolds among k folds } Catalan
  - $k = \frac{n}{3 + \lg n} \Rightarrow 4^k (2n)^k = (8n)^k = 2^{(3+\lg n)\cdot k} = 2^n$
  - $2^n$ M/V strings □

- all M/V strings can be folded by $(4+\varepsilon)n/\lg n$
  - divide string into $\Theta(n/\lg n)$ chunks each of length $(1-\varepsilon)\lg n$
  - $2^{(1-\varepsilon)\lg n} = n^{1-\varepsilon}$ chunk values
    $\Rightarrow$ average chunk repeated $n^{\varepsilon}/\lg n$ times
  - for each chunk value:
    1. fold repetitions together
    2. make their $\Theta(\lg n)$ folds
    3. fix creases messed up in 1
    4. recurse on inverted half

$$n^{1-\varepsilon} \cdot \left(n^{\varepsilon}/\lg n + \lg n + n^{\varepsilon}/\lg n\right) \cdot \underset{\text{geometric series}}{2}$$




**OPEN**: explicit (family of) examples requiring $\Theta(n/\lg n)$ folds?

**OPEN**: complexity of shortest fold sequence of given M/V string?
- known: in EXPTIME

- Let's fold hypars!