6.642 Continuum Electromechanics
Fall 2008

6.642 — Continuum Electromechanics	Fall 2008

Shape of an Interface Bounded by a Vertical Plane Wall

Prof. Markus Zahn MIT OpenCourseWare

Linear Analysis

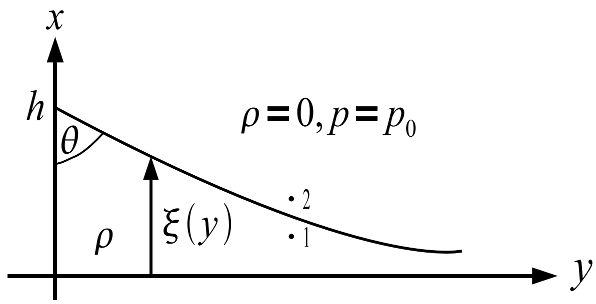


Figure 1: Shape of the surface of a fluid that is in a gravitational field and bounded on one side by a vertical plane wall (Image by MIT OpenCourseWare.)

$$\mathbf{n} = \mathbf{i_x} - \frac{\partial \xi}{\partial y}\mathbf{i_y}$$

$$\mathbf{T_S} = -\gamma(\nabla \cdot \mathbf{n})\mathbf{n} = \gamma \frac{\partial^2 \xi}{\partial y^2}\mathbf{n}$$

$$(p_1 - p_2)\mathbf{n} + T_S\mathbf{n} = 0 \Rightarrow \gamma \frac{\partial^2 \xi}{\partial y^2} + p_1 - p_0 = 0$$

Bernoulli Law: $p_1 + \rho g \xi = p_0 \Rightarrow p_1 - p_0 = -\rho g \xi$

$$\frac{\partial^2 \xi}{\partial y^2} - \frac{\rho g}{\gamma}\xi = 0$$

$$\xi(y) = A_1 e^{-\sqrt{\rho g/\gamma} y} + A_2 e^{+\sqrt{\rho g/\gamma} y}$$

$$\xi(y \to \infty) = 0 \Rightarrow A_2 = 0, \quad \xi(y = 0) = h = A_1$$

$$\xi(y) = h e^{-\sqrt{\rho g/\gamma} y}$$

$$\frac{d\xi}{dy}|_{y=0} = -\sqrt{\frac{\rho g}{\gamma}} h = -\cot\theta \Rightarrow h = \sqrt{\frac{\gamma}{\rho g}}\cot\theta$$

Non-Linear Analysis

$$\mathbf{n} = \frac{\mathbf{i_x} - \frac{\partial \xi}{\partial y}\mathbf{i_y}}{\left[1 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]^{1/2}}$$

$$\mathbf{T_S} = -\gamma(\nabla \cdot \mathbf{n})\mathbf{n} = -\gamma\mathbf{n}\left[\frac{-\frac{\partial^2 \xi}{\partial y^2}}{\left[1 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]^{1/2}} + \frac{\frac{\partial \xi}{\partial y}\frac{\partial \xi}{\partial y}\frac{\partial^2 \xi}{\partial y^2}}{\left[1 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]^{3/2}}\right]$$

$$= +\gamma\mathbf{n}\frac{\frac{\partial^2 \xi}{\partial y^2}}{\left[1 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]^{3/2}}$$

Bernoulli Law: $p_1 - p_0 = -\rho g \xi = -T_S = \dfrac{-\gamma\frac{\partial^2 \xi}{\partial y^2}}{\left[1 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]^{3/2}}$

$$\frac{\frac{\partial^2 \xi}{\partial y^2}}{\left[1 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]^{3/2}} - \frac{\rho g}{\gamma}\xi = 0$$

$$u = \frac{1}{\left[1 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]^{1/2}}$$

$$\frac{du}{d\xi} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi} = \frac{1}{\frac{\partial \xi}{\partial y}}\frac{\partial u}{\partial y}$$

$$= \frac{-\frac{1}{2}2\frac{\partial \xi}{\partial y}\frac{\partial^2 \xi}{\partial y^2}}{\left[1 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]^{3/2}}\frac{1}{\frac{\partial \xi}{\partial y}}$$

$$= \frac{-\frac{\partial^2 \xi}{\partial y^2}}{\left[1 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]^{3/2}}$$

$$\frac{\partial u}{\partial \xi} = -\frac{\rho g}{\gamma}\xi \Rightarrow u = -\frac{\rho g}{\gamma}\frac{\xi^2}{2} + C_1$$

$$\lim_{x\to\infty} \xi = 0, \quad \frac{\partial \xi}{\partial y} = 0$$

$$\lim_{x\to\infty} u = 1 \Rightarrow C_1 = 1$$

$$u = -\frac{\rho g}{2\gamma}\xi^2 + 1$$

$$a^2 = \frac{2\gamma}{\rho g} = \text{capillary length} \Rightarrow u = -\left(\frac{\xi}{a}\right)^2 + 1 = \frac{1}{\left[1 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]^{1/2}}$$

$$\left(\frac{\partial \xi}{\partial y}\right)^2 = \frac{1}{\left(1-\left(\frac{\xi}{a}\right)^2\right)^2} - 1$$

$$= \frac{\left(\frac{\xi}{a}\right)^2\left[2-\left(\frac{\xi}{a}\right)^2\right]}{\left[1-\left(\frac{\xi}{a}\right)^2\right]^2}$$

$$\frac{d\xi}{dy} = \frac{\left(\frac{\xi}{a}\right)\left[2-\left(\frac{\xi}{a}\right)^2\right]^{1/2}}{\left[1-\left(\frac{\xi}{a}\right)^2\right]}$$

$$\frac{\left[1-\left(\frac{\xi}{a}\right)^2\right]d\xi}{\left(\frac{\xi}{a}\right)\left[2-\left(\frac{\xi}{a}\right)^2\right]^{1/2}} = dy$$

$$\frac{y}{a} = \sqrt{2-\left(\frac{\xi}{a}\right)^2} - \frac{\ln\left[\frac{2}{\xi}+\sqrt{\frac{4}{\xi^2}-\frac{2}{a^2}}\right]}{\sqrt{2}} + C$$

$$\xi(y=0) = h \Rightarrow 0 = \sqrt{2-\left(\frac{h}{a}\right)^2} - \frac{\ln\left[\frac{2}{h}+\sqrt{\frac{4}{h^2}-\frac{2}{a^2}}\right]}{\sqrt{2}} + C$$

$$C = -\sqrt{2-\left(\frac{h}{a}\right)^2} + \frac{\ln\left[\frac{2}{h}+\sqrt{\frac{4}{h^2}-\frac{2}{a^2}}\right]}{\sqrt{2}}$$

$$\frac{y}{a} = \sqrt{2-\left(\frac{\xi}{a}\right)^2} - \sqrt{2-\left(\frac{h}{a}\right)^2} - \frac{1}{\sqrt{2}}\ln\left(\frac{\frac{\sqrt{2}a}{3}+\sqrt{\frac{2a^2}{\xi^2}-1}}{\frac{\sqrt{2}a}{h}+\sqrt{\frac{2a^2}{h^2}-1}}\right)$$

$$= \sqrt{2-\left(\frac{\xi}{a}\right)^2} - \sqrt{2-\left(\frac{h}{a}\right)^2} - \frac{1}{\sqrt{2}}\left[\cosh^{-1}\left(\frac{\sqrt{2}a}{3}\right) - \cosh^{-1}\left(\frac{\sqrt{2}a}{h}\right)\right]$$

Identity: $\cosh^{-1} z = \ln[z+(z^2-1)^{1/2}]$

Show[p1, p3, p5, p6, p7, p9, PlotRange → {0, 1}]

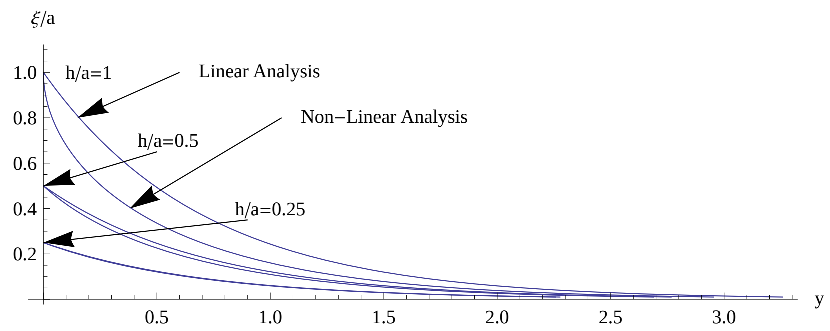


Figure 2: Comparison of linear and nonlinear analyses for wetted wall interfacial shape for various values of h/a (Image by MIT OpenCourseWare.)

Linear analysis: $\theta = \pi/2 - \theta'$

$$h = \sqrt{\frac{\gamma}{\rho g}} \cot\theta = \frac{a}{\sqrt{2}} \frac{\cos\theta}{\sin\theta} = \frac{a}{\sqrt{2}} \frac{\cos\left(\frac{\pi}{2} - \theta'\right)}{\sin\left(\frac{\pi}{2} - \theta'\right)} \approx \frac{a}{\sqrt{2}} \frac{\sin\theta'}{\cos\theta'} \approx \frac{a}{\sqrt{2}} \frac{\theta'}{1 - \frac{\theta'^2}{2}} \approx \frac{a\theta'}{\sqrt{2}}$$

$$h \approx \frac{a\theta'}{\sqrt{2}}$$

Nonlinear analysis in limit $\theta \approx \frac{\pi}{2} - \theta'$:

$$\frac{h}{a} = \left[1 - \sin\left(\frac{\pi}{2} - \theta'\right)\right]^{1/2} \approx (1 - \cos\theta')^{1/2} = \left[1 - \left(1 - \frac{\theta'^2}{2}\right)\right]^{1/2} = \left(\frac{\theta'^2}{2}\right)^{1/2} = \frac{\theta'}{\sqrt{2}}$$

$$\frac{h}{a} \approx \frac{\theta'}{\sqrt{2}} \text{ (in agreement with linear analysis)}$$

Mathematica Program for Interfacial Shape Analyses

Integrate[−(1 − (xi/a)^2)/((xi/a) ∗ (2 − (xi/a)^2)^(1/2)), xi]

$$a\left(-\sqrt{2-\frac{\mathrm{xi}^2}{a^2}}-\frac{\mathrm{Log}\left[\sqrt{2}\mathrm{xi}\right]}{\sqrt{2}}+\frac{\mathrm{Log}\left[-2\left(2+\sqrt{4-\frac{2\mathrm{xi}^2}{a^2}}\right)\right]}{\sqrt{2}}\right)$$

$$\mathbf{f1[xi\_, h\_] = -\left(\sqrt{2-\frac{xi^2}{a^2}}-\frac{Log\left[Sqrt[2]*a/xi+\sqrt{4*a\text{^}2/xi\text{^}2-1}\right]}{\sqrt{2}}-\sqrt{2-h\text{^}2/a^2}+\frac{Log\left[Sqrt[2]*a/h+\sqrt{4*a\text{^}2/h\text{^}2-1}\right]}{\sqrt{2}}\right)}$$

$$\sqrt{2-\frac{h^2}{a^2}}-\sqrt{2-\frac{\mathrm{xi}^2}{a^2}}-\frac{\mathrm{Log}\left[\sqrt{-1+\frac{4a^2}{h^2}}+\frac{\sqrt{2}a}{h}\right]}{\sqrt{2}}+\frac{\mathrm{Log}\left[\sqrt{-1+\frac{4a^2}{\mathrm{xi}^2}}+\frac{\sqrt{2}a}{\mathrm{xi}}\right]}{\sqrt{2}}$$

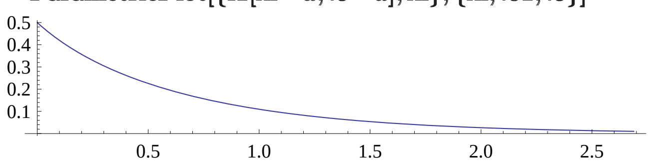
$$\mathbf{f2[xi\_, h\_] = -\left(\sqrt{2-\frac{xi^2}{a^2}}-\sqrt{2-h\text{^}2/a^2}-ArcCosh[Sqrt[2]*a/xi]/Sqrt[2]+ArcCosh[Sqrt[2]*a/h]/Sqrt[2]\right)}$$

$$\sqrt{2-\frac{h^2}{a^2}}-\sqrt{2-\frac{\mathrm{xi}^2}{a^2}}-\frac{\mathrm{ArcCosh}\left[\frac{\sqrt{2}a}{h}\right]}{\sqrt{2}}+\frac{\mathrm{ArcCosh}\left[\frac{\sqrt{2}a}{\mathrm{xi}}\right]}{\sqrt{2}}$$

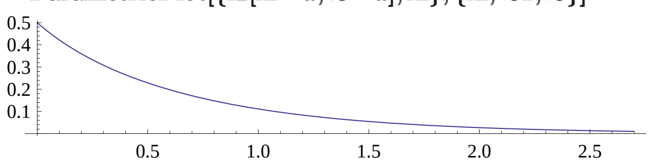
f3[xi_, h_] = −Log[xi/h]/Sqrt[2]

$$-\frac{\mathrm{Log}\left[\frac{\mathrm{xi}}{h}\right]}{\sqrt{2}}$$

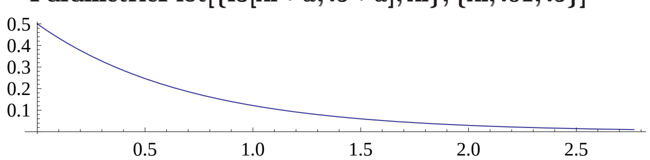
p1 = ParametricPlot[{f1[xi ∗ a, .5 ∗ a], xi}, {xi, .01, .5}]



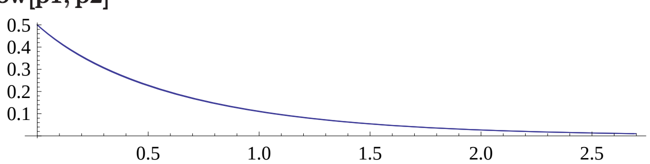
p2 = ParametricPlot[{f2[xi ∗ a, .5 ∗ a], xi}, {xi, .01, .5}]



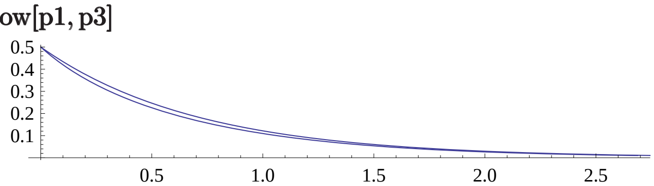
p3 = ParametricPlot[{f3[xi ∗ a, .5 ∗ a], xi}, {xi, .01, .5}]



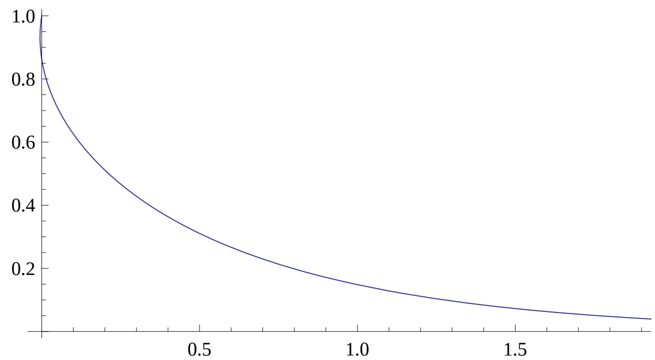
Show[p1, p2]



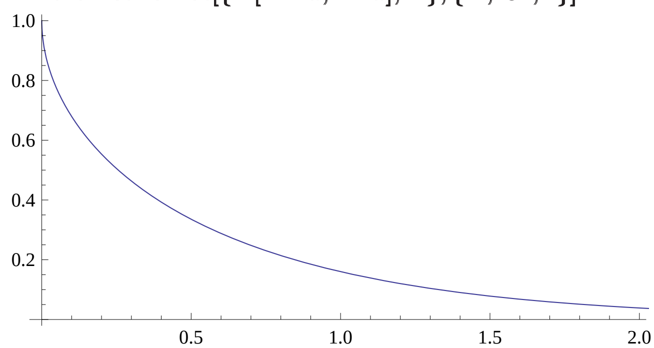
Show[p1, p3]



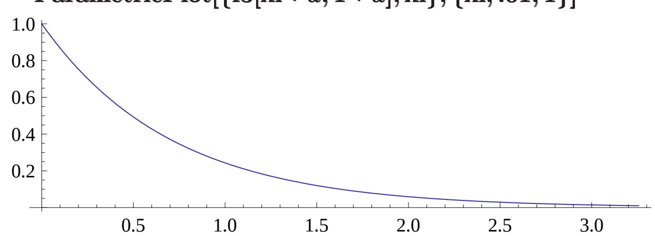
p4 = ParametricPlot[{f1[xi ∗ a, 1 ∗ a], xi}, {xi, .01, 1}]



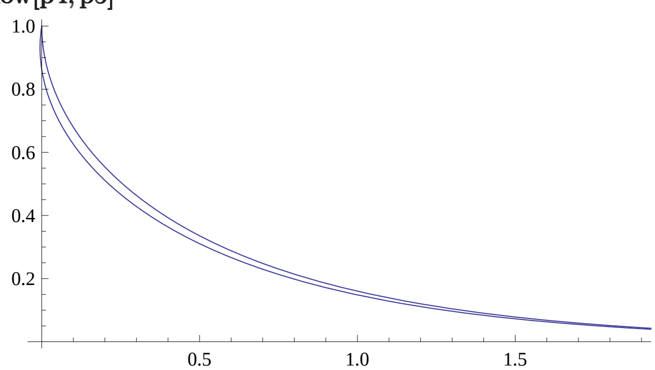
p5 = ParametricPlot[{f2[xi * a, 1 * a], xi}, {xi, .01, 1}]



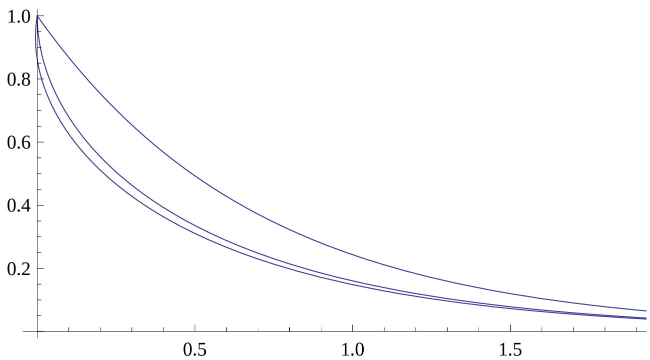
p6 = ParametricPlot[{f3[xi * a, 1 * a], xi}, {xi, .01, 1}]



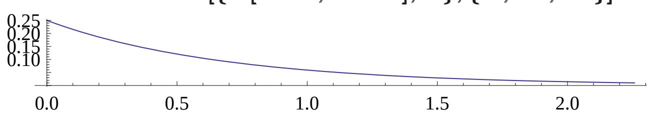
Show[p4, p5]



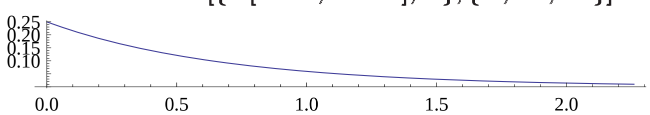
Show[p4, p5, p6]



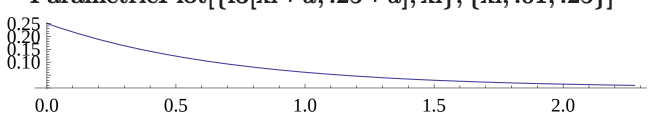
p7 = ParametricPlot[{f1[xi ∗ *a*, .25 ∗ *a*], xi}, {xi, .01, .25}]



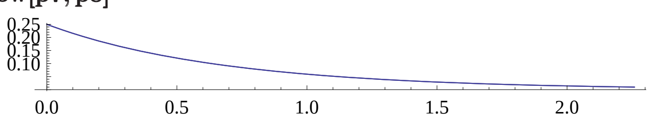
p8 = ParametricPlot[{f2[xi ∗ *a*, .25 ∗ *a*], xi}, {xi, .01, .25}]



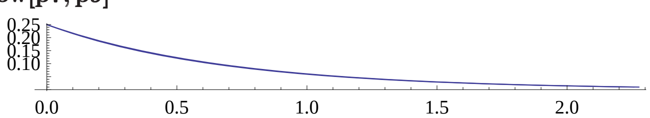
p9 = ParametricPlot[{f3[xi ∗ *a*, .25 ∗ *a*], xi}, {xi, .01, .25}]



Show[p7, p8]



Show[p7, p9]



Show[p1, p3, p5, p6, p7, p9, PlotRange → {0, 1}]

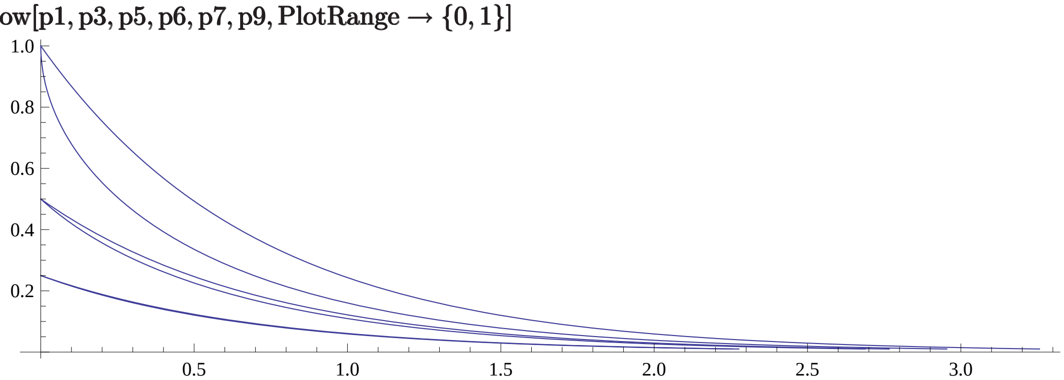
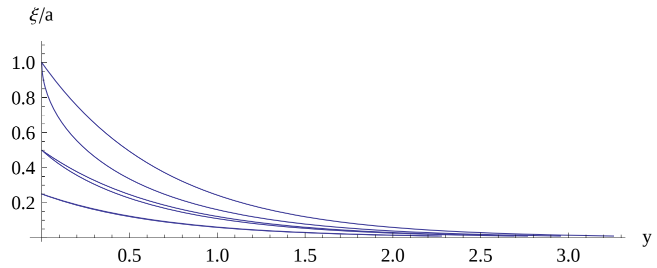


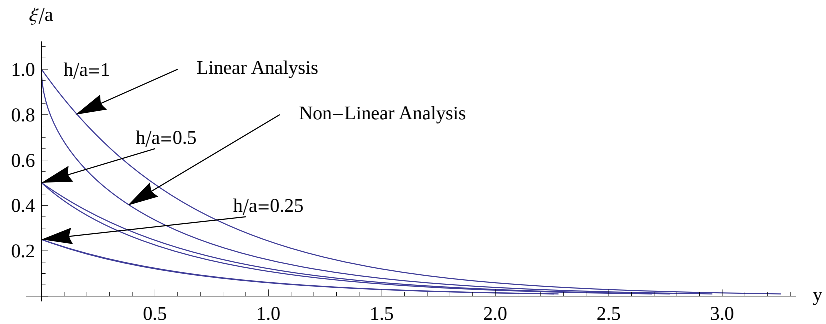
Fig1 = Show[p1, p3, p5, p6, p7, p9, PlotRange → {0, 1.1}, AxesLabel → {"y", "ξ/a"}]



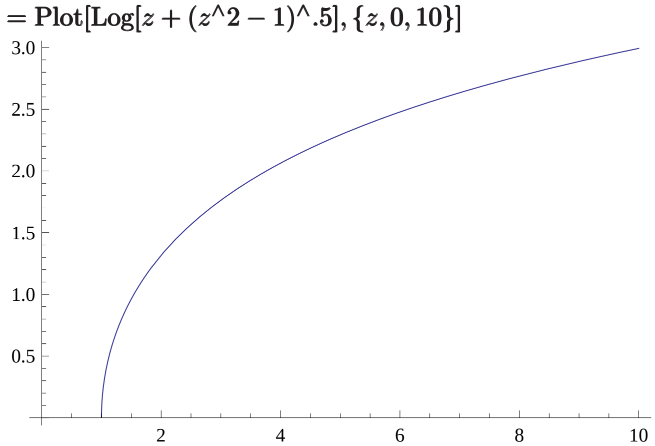
arrows = {Graphics[Arrow[{{.6, 1.}, {.15, .8}}]], Graphics[Arrow[{{.5, .65}, {0, .5}}]], Graphics[Arrow[{{.9, .35}, {0, .25}}]], Graphics[Arrow[{{1.05, 0.8}, {.38, .4}}]]}

{ , , , }

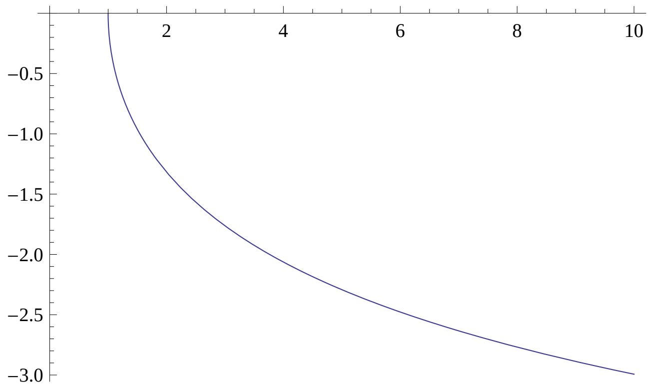
Show[Fig1, Graphics[Text["h/a=1", {.2, 1.0}]], Graphics[Text["h/a=0.5", {.55, .7}]], Graphics[Text["h/a=0.25", {1., .4}]], Graphics[Text["Linear Analysis", {.95, 1.}]], Graphics[Text["Non-Linear Analysis", {1.5, .8}]], arrows]



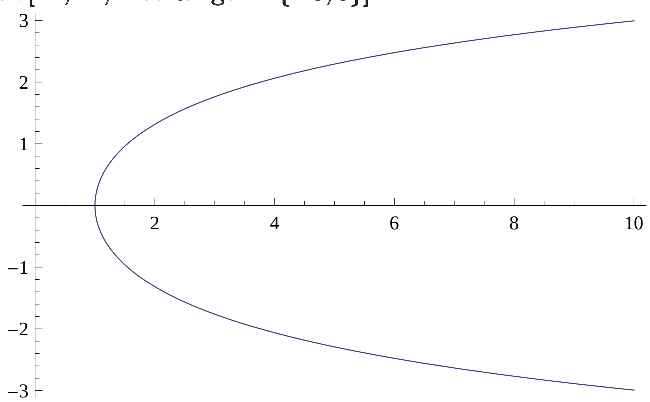
h1 = Plot[Log[z + (z^2 − 1)^.5], {z, 0, 10}]



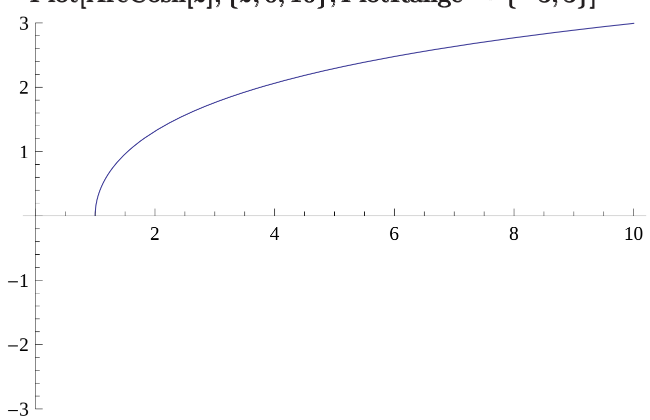
h2 = Plot[Log[z − (z^2 − 1)^.5], {z, 0, 10}]



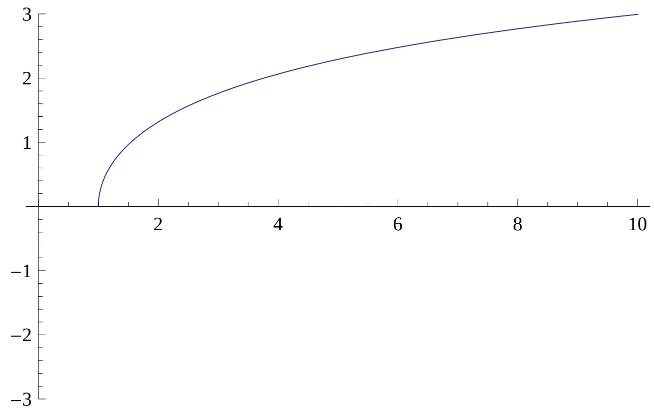
Show[h1, h2, PlotRange → {−3, 3}]



h3 = Plot[ArcCosh[z], {z, 0, 10}, PlotRange → {−3, 3}]



h3 = Plot[ArcCosh[z], {z, 0, 10}, PlotRange → {−3, 3}]


3
2
1
−1
−2
−3
2
4
6
8
10