6.642 Continuum Electromechanic
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Problem Set 8 - Solutions

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Problem 8.18.2

The basic equations for the magnetizable but insulating inhomogeneous fluid are

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p - \rho g \mathbf{i_x} - \frac{1}{2}H^2\nabla\mu, \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2}$$

$$\nabla\cdot\mu\mathbf{h} = 0, \tag{3}$$

$$\nabla\times\mathbf{h} = 0, \tag{4}$$

$$\frac{D\mu}{Dt} = 0, \tag{5}$$

$$\frac{D\rho}{Dt} = 0, \tag{6}$$

where $\mathbf{H} = H_s(x)\mathbf{i_z} + \mathbf{h}$.

In view of Eq. 4, $\mathbf{h} = -\nabla\chi$. This means that $\hat{h}_z = jk_z\hat{\chi}$ and for the present purposes it is more convenient to use $\hat{h}_z$ as a scalar "potential"

$$\hat{h}_x = -\frac{1}{jk_z}D\hat{h}_z; \quad \hat{h}_y = \frac{k_y}{k_z}\hat{h}_z. \tag{7}$$

With the definitions $\mu = \mu_s(x) + \mu'$ and $\rho = \rho_s(x) + \rho'$, Eqs. 5 and 6 link the perturbations in properties to the fluid displacement

$$\hat{\mu} = -\frac{\hat{v}_x D\mu_s}{j\omega}; \quad \hat{\rho} = -\frac{\hat{v}_x D\hat{\rho}_s}{j\omega}. \tag{8}$$

Thus, with the use of Eq. 8a and Eqs. 7, the linearized version of Eq. 3 is

$$D(\mu_s D\hat{h}_z) = k^2\mu_s\hat{h}_z + j\frac{k_z^2}{\omega}H_s(D\mu_s)\hat{v}_x; \quad k^2 \equiv k_y^2 + k_z^2, \tag{9}$$

and this represents the magnetic field, given the mechanical deformation.

To represent the mechanics, Eq. 2 is written in terms of complex amplitudes:

$$D\hat{v}_x = jk_y\hat{v}_y + jk_z\hat{v}_z, \tag{10}$$

and, with the use of Eq. 8b, the x component of Eq. 1 is written in the linearized form

$$\left[\omega^2\rho_s + gD\rho_s + \frac{1}{2}H_s^2D^2\mu_s\right]\hat{\vartheta}_x + \frac{1}{2}H_s^2(D\mu_s)D\hat{v}_x - j\omega H_s(D\mu_s)\hat{h}_z = j\omega D\hat{p}. \tag{11}$$

Similarly, the y and z components of Eq. 1 become

$$j\omega\rho_s\hat{v}_y = jk_y\hat{p} - \frac{1}{2}\frac{k_y}{\omega}H_s^2(D\mu_s)\hat{v}_x, \tag{12}$$

$$j\omega\rho_s\hat{v}_z = jk_z\hat{p} - \frac{1}{2}\frac{k_z}{\omega}H_s^2(D\mu_s)\hat{v}_x. \tag{13}$$

With the objective of making $\hat{v}_x$ a scalar function representing the mechanics, these last two expressions are solved for $\hat{v}_y$ and $\hat{v}_z$ and substituted into Eq. 10:

$$\omega\rho_s D\hat{v}_x = jk^2\hat{p} - \frac{1}{2}\frac{k^2}{\omega}H_s^2(D\mu_s)\hat{v}_x. \tag{14}$$

This expression is then solved for $\hat{p}$, and the derivative taken with respect to x. This derivative can then be used to eliminate the pressure from Eq. 11:

$$D[\rho_s(D\hat{v}_x)] - k^2\left[\rho_s - \frac{\mathcal{N}}{\omega^2}\right]\hat{v}_x + j\frac{k^2 H_s(D\mu_s)}{\omega}\hat{h}_z = 0, \tag{15}$$

$$\mathcal{N} \equiv -gD\rho_s + \frac{1}{2}(D\mu_s)D(H_s^2) \cong -gD\rho_s.$$

Equations 9 and 15 comprise the desired relations.

In an imposed field approximation where $H_s = H_0 =$ constant and the properties have the profiles $\rho_s = \rho_m \exp\beta x$ and $\mu_s = \mu_m \exp\beta x$, Eqs. 9 and 15 become

$$\left[L + \frac{k^2\mathcal{N}}{\rho_s\omega^2}\right]\hat{v}_x + \left[\frac{jk^2 H_0\beta\mu_m}{\rho_m\omega}\right]\hat{h}_z = 0, \tag{16}$$

$$[L]\hat{h}_z + \left[\frac{jk_z^2 H_0\beta}{\omega}\right]\hat{v}_x = 0, \tag{17}$$

where $L \equiv D^2 + \beta D - k^2$.

For these constant coefficient equations, solutions take the form $\exp\gamma x$ and $L \to \gamma^2 + \beta\gamma - k^2$. From Eqs. 16 and 17 it follows that

$$L^2 + \frac{k^2\mathcal{N}}{\rho_s\omega^2}L + \frac{k^2k_z^2}{\omega^2}\frac{H_0^2\beta^2\mu_m}{\rho_m} = 0. \tag{18}$$

Solution for L results in

$$L = a \pm b; \quad a \equiv \frac{g\beta k^2}{2\omega^2}; \quad b \equiv \left[\left(\frac{g\beta k^2}{2\omega^2}\right)^2 - \left(\frac{kk_z}{\omega}\frac{H_0\beta}{\sqrt{\rho_m/\mu_m}}\right)^2\right]^{\frac{1}{2}}. \tag{19}$$

From the definition of L, the γ's representing the x dependence follow as

$$\gamma = -\frac{\beta}{2} \pm c_\pm; \quad c_\pm \equiv \left[\left(\frac{\beta}{2}\right)^2 + k^2 + a \pm b\right]^{\frac{1}{2}}. \tag{20}$$

In terms of these γ's,

$$\hat{v}_x = e^{-\frac{\beta}{2}x}\left[\hat{A}_1 e^{c_+x} + \hat{A}_2 e^{-c_+x} + \hat{A}_3 e^{c_-x} + \hat{A}_4 e^{-c_-x}\right]. \tag{21}$$

The corresponding $\hat{h}_z$ is written in terms of these same coefficients with the help of Eq. 17:

$$\hat{h}_z = -j\frac{k_z^2 H_0 \beta}{\omega}\left[\frac{\hat{A}_1 e^{c_+ x}}{a+b} + \frac{\hat{A}_2 e^{-c_+ x}}{a+b} + \frac{\hat{A}_3 e^{c_- x}}{a-b} + \frac{\hat{A}_4 e^{-c_- x}}{a-b}\right] e^{-\frac{\beta}{2}x}. \tag{22}$$

Thus, the four boundary conditions require that

$$\begin{bmatrix} e^{c_+\ell} & e^{-c_+\ell} & e^{c_-\ell} & e^{-c_-\ell} \\ 1 & 1 & 1 & 1 \\ \frac{e^{c_+\ell}}{a+b} & \frac{e^{-c_+\ell}}{a+b} & \frac{e^{c_-\ell}}{a-b} & \frac{e^{-c_-\ell}}{a-b} \\ \frac{1}{a+b} & \frac{1}{a+b} & \frac{1}{a-b} & \frac{1}{a-b} \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \\ \hat{A}_4 \end{bmatrix} = 0. \tag{23}$$

This determinant is easily reduced by first subtracting the second and fourth columns from the first and third respectively and then expanding by minors:

$$\sinh(c_+\ell)\sinh(c_-\ell)\frac{2b}{a^2-b^2} = 0. \tag{24}$$

Thus, eigenmodes are $c_+\ell = jn\pi$ and $c_-\ell = jn\pi$. The eigenfrequencies follow from Eqs. 19 and 20:

$$\omega_n^2 = \frac{k^2 k_z^2 H_0^2 \beta^2 \mu_m}{K_n^4 \rho_m} - \frac{g\beta k^2}{K_n^2}; \quad K_n \equiv \left(\frac{n\pi}{\ell}\right)^2 + \left(\frac{\beta}{2}\right)^2 + k^2. \tag{25}$$

For perturbations with peaks and valleys running perpendicular to the imposed fields, the magnetic field stiffens the fluid. Internal electromechanical waves propagate along the lines of magnetic field intensity. If the fluid were confined between parallel plates in the $x-z$ planes, so that the fluid were indeed forced to undergo only two dimensional motions, the field could be used to balance a heavy fluid on top of a light one.... to prevent the gravitational form of Rayleigh-Taylor instability. However, for perturbations with hills and valleys running parallel to the imposed field, the magnetic field remains undisturbed, and there is no magnetic restoring force to prevent the instability. The role of the magnetic field, here in the context of an internal coupling, is similar to that for the hydromagnetic system described in Sec. 8.12 where interchange modes of instability for a surface coupled system were found.

The electric polarization analogue to this configuration might be as shown in Fig. 8.11.1, but with a smooth distribution of ϵ and ρ in the x direction.

Solution to Problem 8.18.2 in Melcher, James R. *Solutions Manual for Continuum Electromechanics*, 1982, pp. 8.52-8.55.

Problem 8.18.3

Starting with Eqs. 9 and 15 from Prob. 8.18.2, multiply the first by $\hat{h}_z^*$ and integrate from 0 to ℓ.

$$\int_0^\ell \hat{h}_z^* D(\mu_s D\hat{h}_z)dx - \int_0^\ell k^2\mu_s \hat{h}_z \hat{h}_z^* dx - j\int_0^\ell \frac{k_z^2}{\omega}(D\mu_s)H_s \hat{v}_x \hat{h}_z^* dx = 0. \tag{26}$$

Integration of the first term by parts and use of the boundary conditions on $\hat{h}_z$ gives integrals on the left that are positive definite:

$$-\int_0^\ell \mu_s (D\hat{h}_z)(D\hat{h}_z)^* dx - k^2\int_0^\ell \mu_s \hat{h}_z \hat{h}_z^* dx - j\frac{k_z^2}{\omega}\int_0^\ell (D\mu_s)H_s \hat{v}_x \hat{h}_z^* dx = 0. \tag{27}$$

In summary,

$$I_1 = -j\frac{k_z^2}{\omega}\hat{I}_4^*; \quad I_1 \equiv \int_0^\ell [\mu_s |D\hat{h}_z|^2 + k^2\mu_s |\hat{h}_z|^2]dx, \quad \hat{I}_4 \equiv \int_0^\ell H_s (D\mu_s)\hat{v}_x^* \hat{h}_z dx. \tag{28}$$

Now, multiply Eq. 15 from Prob. 8.18.2 by $\hat{v}_x^*$ and integrate:

$$\int_0^\ell \hat{v}_x^* D\rho_s(D\hat{v}_x)dx - k^2\int_0^\ell \rho_s\hat{v}_x\hat{v}_x^* dx + \frac{k^2}{\omega^2}\int_0^\ell \mathcal{N}\hat{v}_x\hat{v}_x^* dx + j\frac{k^2}{\omega}\int_0^\ell H_s D\mu_s\hat{v}_x^*\hat{h}_z dx = 0. \tag{29}$$

Integration of the first term by parts and the boundary conditions on $\hat{v}_x$ gives

$$-\int_0^\ell \rho_s D\hat{v}_x D\hat{v}_x^* dx - k^2\int_0^\ell \rho_s\hat{v}_x\hat{v}_x^* dx + \frac{k^2}{\omega^2}\int_0^\ell \mathcal{N}\hat{v}_x\hat{v}_x^* dx + \frac{jk^2}{\omega}\int_0^\ell H_s(D\mu_s)\hat{v}_x^*\hat{h}_z dx = 0, \tag{30}$$

and this expression takes the form

$$I_2 - \frac{I_3}{\omega^2} = j\frac{k^2}{\omega}\hat{I}_4; \quad I_2 \equiv \int_0^\ell (\rho_s|D\hat{v}_x|^2 + k^2\rho_s|\hat{v}_x|^2)dx; \quad I_3 = \int_0^\ell k^2\mathcal{N}|\hat{v}_x|^2 dx. \tag{31}$$

Multiplication of Eq. 28 by Eq. 31 results in yet another positive definite quantity

$$I_1 I_2 - \frac{I_1 I_3}{\omega^2} = \frac{k^2 k_z^2}{\omega^2}|\hat{I}_4|^2, \tag{32}$$

and this expression can be solved for the frequency

$$\omega^2 = \frac{k^2 k_z^2|\hat{I}_4|^2 + I_1 I_3}{I_1 I_2}. \tag{33}$$

Because the terms on the right are real, it follows that either the eigenfrequencies are real or they represent modes that grow and decay without oscillation. Thus, the search for eigenfrequencies in the general case can be restricted to the real and imaginary axes of the s plane.

Note that a sufficient condition for stability is $\mathcal{N} > 0$, because that insures that I_3 is positive definite.