6.642 Continuum Electromechanics
Fall 2008

**6.642 — Continuum Electromechanics** Fall 2008

# Problem Set 4 - Solutions

Prof. Markus Zahn MIT OpenCourseWare

## Problem 1

### a)

$$\nabla \times \mathbf{H} = 0 \Rightarrow \mathbf{H} = -\nabla \Phi$$

$$\nabla \cdot \mathbf{H} = 0 \Rightarrow \nabla^2 \Phi = 0$$

$$r > R \qquad \Phi(r,\theta) = \left(Dr + \frac{C}{r^2}\right)\cos\theta$$

$$\mathbf{H} = -\nabla\Phi = -\left(\frac{\partial \Phi}{\partial r}\mathbf{i_r} + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial \Phi}{\partial \phi}}\mathbf{i}_\phi\right)$$

$$= -\left[\left(D - \frac{2C}{r^3}\right)\cos\theta\mathbf{i_r} - \frac{1}{r}\left(Dr + \frac{C}{r^2}\right)\sin\theta\mathbf{i}_\theta\right] \quad r > R$$

B.C. $\mathbf{H}(r \to \infty, \theta) = H_0\mathbf{i_z} = H_0(\mathbf{i_r}\cos\theta - \mathbf{i}_\theta\sin\theta) = -(D\cos\theta\mathbf{i_r} - D\sin\theta\mathbf{i}_\theta)$

$$D = -H_0$$

$$H_\theta(r = R, \theta) = 0 = \frac{1}{R}\left(DR + \frac{C}{R^2}\right)\sin\theta \Rightarrow C = -DR^3 = H_0R^3$$

$$\mathbf{H} = H_0\left[\left(1 + \frac{2R^3}{r^3}\right)\cos\theta\mathbf{i_r} - \left(1 - \frac{R^3}{r^3}\right)\sin\theta\mathbf{i}_\theta\right] \quad r > R$$

$$\mathbf{B} = \mu_0\mathbf{H}$$

### b)

$$\frac{dr}{rd\theta} = \frac{\mu_0 H_r}{\mu_0 H_\theta} = \frac{\left(1 + \frac{2R^3}{r^3}\right)\cos\theta}{-\left(1 - \frac{R^3}{r^3}\right)\sin\theta}$$

$$\frac{-\left(1 - \frac{R^3}{r^3}\right)dr}{r\left(1 + \frac{2R^3}{r^3}\right)} = \frac{\cos\theta}{\sin\theta}d\theta$$

$$\int \frac{\cos\theta}{\sin\theta}d\theta = \ln(\sin\theta) + C_1$$

$$\int \frac{\left(1 - \frac{R^3}{r^3}\right)dr}{r\left(1 + \frac{2R^3}{r^3}\right)} = \int \frac{(r^3 - R^3)dr}{r(r^3 + 2R^3)} = -\frac{\ln r}{2} + \frac{1}{2}\ln(r^3 + 2R^3) + C_2$$

$$\ln(\sin\theta) + C_1 = \frac{\ln r}{2} - \frac{1}{2}\ln(r^3 + 2R^3) + C_2$$

$$\ln[\sin^2\theta(r^3 + 2R^3)/r] = 2(C_2 - C_1)$$

$$\sin^2\theta(r^3+2R^3)/r = \text{constant} \Rightarrow \sin^2\theta\left(\frac{1}{2}\left(\frac{r}{R}\right)^2+\frac{R}{r}\right) = \text{constant}$$

equation of magnetic field lines for $r > R$

## c)

From Melcher's Continuum Electromechanics, Appendix A with $\mathbf{A}(r,\theta) = A(r,\theta)\mathbf{i}_\phi$.

$$\begin{aligned}\nabla^2 A(r,\theta))\mathbf{i}_\phi &= \left(\nabla^2 A_\phi - \frac{A_\phi}{r^2\sin^2\theta}\right)\mathbf{i}_\phi \\ &= \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\phi}{\partial\theta}\right)-\frac{A_\phi}{r^2\sin^2\theta}\right]\mathbf{i}_\phi \\ &= 0\end{aligned}$$

$$\begin{aligned}\nabla\times\mathbf{A}(r,\theta) = \nabla\times(A(r,\theta)\mathbf{i}_\phi) &= \mathbf{i_r}\frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta A_\phi)}{\partial\theta}\right]-\mathbf{i}_\theta\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} \\ &= B_r\mathbf{i_r}+B_\theta\mathbf{i}_\theta\end{aligned}$$

$$B_\theta = -\mu_0H_0\left(1-\frac{R^3}{r^3}\right)\sin\theta = -\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}$$

$$\frac{1}{\sin\theta}\frac{\partial(rA_\phi)}{\partial r} = \mu_0H_0\left(r-\frac{R^3}{r^2}\right)\Rightarrow rA_\phi = \mu_0H_0\left(\frac{r^2}{2}+\frac{R^3}{r}\right)\sin\theta+f(\theta)$$

$$\begin{aligned}B_r = +\mu_0H_0\left(1+\frac{2R^3}{r^3}\right)\cos\theta &= \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\theta} \\ &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\mu_0H_0\sin^2\theta\left(\frac{r}{2}+\frac{R^3}{r^2}\right)+\frac{\sin\theta f(\theta)}{r}\right]\end{aligned}$$

$$\begin{aligned}\mu_0H_0\left(1+\frac{2R^3}{r^3}\right)\cos\theta &= \frac{1}{r\sin\theta}\left[\mu_0H_0\left(\frac{r}{2}+\frac{R^3}{r^2}\right)2\sin\theta\cos\theta+\frac{1}{r}\frac{d}{d\theta}(\sin\theta f(\theta))\right] \\ &= \mu_0H_0\left(\frac{r}{2}+\frac{R^3}{r^2}\right)\frac{2\cos\theta}{r}+\frac{1}{r^2\sin\theta}\frac{d}{d\theta}(\sin\theta f(\theta)) \\ &= \mu_0H_0\left(1+\frac{2R^3}{r^3}\right)\cos\theta+\frac{1}{r^2\sin\theta}\underbrace{\frac{d}{d\theta}(\sin\theta f(\theta))}_{0\Rightarrow f(\theta)=0}\end{aligned}$$

$$A_\phi = \mu_0H_0\left(\frac{r}{2}+\frac{R^3}{r^2}\right)\sin\theta$$

## d)

$$\nabla^2A_\phi(r,\theta)\mathbf{i}_\phi = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\phi}{\partial\theta}\right)-\frac{A_\phi}{r^2\sin^2\theta}\right]\mathbf{i}_\phi$$

$$\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\phi}{\partial r}\right) &= \frac{\mu_0H_0\sin\theta}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{1}{2}-\frac{2R^3}{r^3}\right)\right] \\ &= \frac{\mu_0H_0\sin\theta}{r^2}\left(r+\frac{2R^3}{r^2}\right)\end{aligned}$$

$$\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial A_\phi}{\partial \theta}\right) = \frac{\mu_0 H_0 \left(\frac{r}{2} + \frac{R^3}{r^2}\right)}{r^2 \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta\cos\theta)$$
$$= \frac{\mu_0 H_0 \left(\frac{r}{2} + \frac{R^3}{r^2}\right)}{r^2 \sin\theta} \cos 2\theta$$

$$\frac{A_\phi}{r^2 \sin^2\theta} = \frac{\mu_0 H_0}{r^2 \sin^2\theta}\left(\frac{r}{2} + \frac{R^3}{r^2}\right)\sin\theta = \frac{\mu_0 H_0 \left(\frac{r}{2} + \frac{R^3}{r^2}\right)}{r^2 \sin\theta}$$

$$\nabla^2 A_\phi(r,\theta)\mathbf{i}_\phi = \mathbf{i}_\phi\left(\frac{\mu_0 H_0 \sin\theta}{r^2}\right)\left[r + \frac{2R^3}{r^2} + \left(\frac{r}{2} + \frac{R^3}{r^2}\right)\frac{\cos 2\theta}{\sin^2\theta} - \frac{\frac{r}{2} + \frac{R^3}{r^2}}{\sin^2\theta}\right]$$
$$= \mathbf{i}_\phi\left(\frac{\mu_0 H_0 \sin\theta}{r^2}\right)\left[r + \frac{2R^3}{r^2} + \left(\frac{r}{2} + \frac{R^3}{r^2}\right)\underbrace{\frac{\cos 2\theta - 1}{\sin^2\theta}}_{-2}\right]$$
$$= 0$$

## e)

Magnetic field line: $r\sin\theta A_\phi(r,\theta) = \text{constant}$

$r\sin\theta\mu_0 H_0\left(\frac{r}{2} + \frac{R^3}{r^2}\right)\sin\theta = \text{constant}$

$\sin^2\theta\left(\frac{1}{2}\left(\frac{r}{R}\right)^2 + \frac{R}{r}\right) = \text{constant} = \frac{3}{2}$ when $r = R, \theta = \frac{\pi}{2}$

$x = r\sin\theta\cos\phi = D$ for $\phi = 0$ at $r = \infty, \theta = \pi$

$\lim_{r\to\infty,\theta=\pi} r\sin\theta = D$

$\frac{1}{2}\frac{D^2}{R^2} = \frac{3}{2} \Rightarrow D = \sqrt{3}R$

**f)**

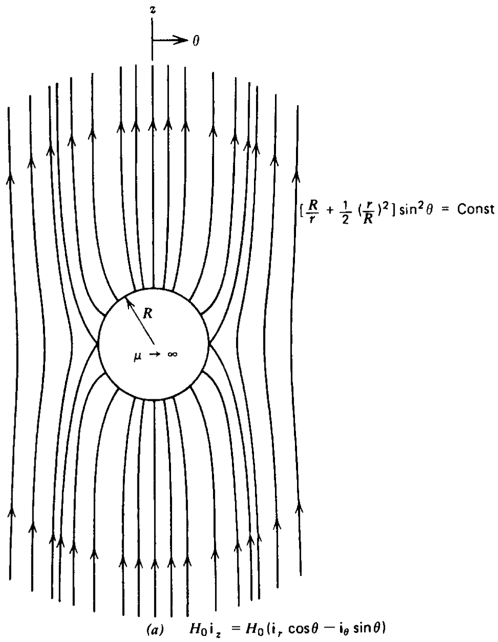


Figure 5-25 Magnetic field lines about an (*a*) infinitely permeable and (*b*) perfectly conducting sphere in a uniform magnetic field.

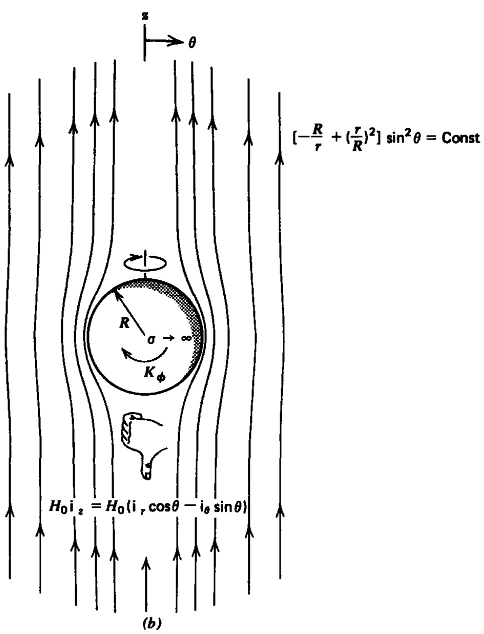


Figure 5-25

# Problem 2

$$\mathbf{F} = (\mathbf{P}\cdot\nabla)\mathbf{E} + \rho_f\mathbf{E}, \quad \mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E}, \quad \rho_f = \nabla\cdot\mathbf{D}$$

$$\nabla \times \mathbf{E} = 0 \Rightarrow \frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}$$

$$\mathbf{F} = (\mathbf{D} - \epsilon_0 \mathbf{E}) \cdot \nabla \mathbf{E} + (\nabla \cdot \mathbf{D})\mathbf{E}$$

$$\begin{aligned}
F_i &= (D_j - \epsilon_0 E_j)\frac{\partial E_i}{\partial x_j} + \frac{\partial D_j}{\partial x_j} E_i \\
&= D_j \frac{\partial E_i}{\partial x_j} + E_i \frac{\partial D_j}{\partial x_j} - \underbrace{\epsilon_0 E_j \frac{\partial E_i}{\partial x_j}}_{\epsilon_0 E_j \frac{\partial E_j}{\partial x_i}} \\
&= \frac{\partial}{\partial x_j}(D_j E_i) - \frac{\epsilon_0}{2}\frac{\partial}{\partial x_i}(E_j E_j) \\
&= \frac{\partial}{\partial x_j}\left[(D_j E_i) - \frac{1}{2}\delta_{ij}\epsilon_0 E_k E_k\right], \quad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \\
&= \frac{\partial T_{ij}}{\partial x_j} \Rightarrow T_{ij} = D_j E_i - \frac{1}{2}\delta_{ij}\epsilon_0 E_k E_k
\end{aligned}$$

## Problem 3

$$\mathbf{F} = \mu_0(\mathbf{M} \cdot \nabla)\mathbf{H} + \mathbf{J} \times \mu_0 \mathbf{H}, \quad \mu_0 \mathbf{M} = \mathbf{B} - \mu_0 \mathbf{H}, \quad \nabla \times \mathbf{H} = \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0$$

$$(\nabla \times \mathbf{H}) \times \mathbf{H} = (\mathbf{H} \cdot \nabla)\mathbf{H} - \frac{1}{2}\nabla(\mathbf{H} \cdot \mathbf{H})$$

$$\begin{aligned}
\mathbf{F} &= (\mathbf{B} - \mu_0 \mathbf{H}) \cdot \nabla \mathbf{H} + \mu_0 (\nabla \times \mathbf{H}) \times \mathbf{H} \\
&= \mathbf{B} \cdot \nabla \mathbf{H} - \mu_0 (\mathbf{H} \cdot \nabla)\mathbf{H} + \mu_0((\mathbf{H} \cdot \nabla)\mathbf{H} - \frac{1}{2}\nabla(\mathbf{H} \cdot \mathbf{H})) \\
&= (\mathbf{B} \cdot \nabla)\mathbf{H} - \frac{\mu_0}{2}\nabla(\mathbf{H} \cdot \mathbf{H})
\end{aligned}$$

$$\begin{aligned}
F_i &= B_j \frac{\partial H_i}{\partial x_j} - \frac{\mu_0}{2}\frac{\partial (H_k H_k)}{\partial x_i} \\
&= \frac{\partial (B_j H_i)}{\partial x_j} - H_i \underbrace{\frac{\partial B_j}{\partial x_j}}_{\nabla \cdot \mathbf{B} = 0} - \frac{\partial}{\partial x_i}\left(\frac{\mu_0}{2} H_k H_k\right) \\
&= \frac{\partial (B_j H_i)}{\partial x_j} - \delta_{ij}\frac{\partial}{\partial x_j}\left(\frac{\mu_0}{2} H_k H_k\right) \Rightarrow T_{ij} = H_i B_j - \frac{1}{2}\delta_{ij}\mu_0 H_k H_k
\end{aligned}$$