6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

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Quiz 1 - Solutions - Spring 2009

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Problem 1

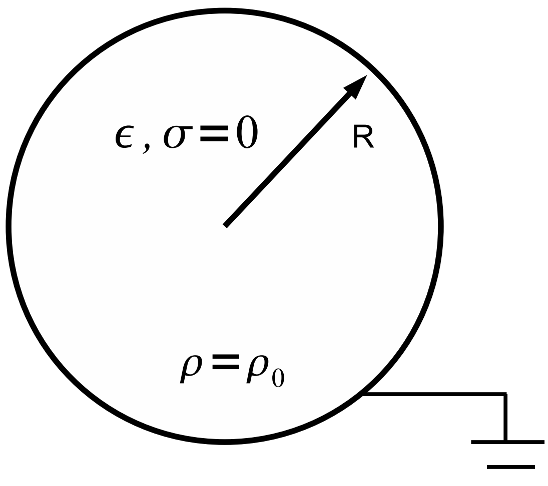

Figure 1: A diagram of a perfectly conducting hollow sphere filled with a perfectly insulating dielectric with a uniform distribution of volume charge (Image by MIT OpenCourseWare).

A perfectly conducting hollow sphere of radius R is filled with a perfectly insulating dielectric ($\sigma = 0$) with a uniform distribution of volume charge:

$$\rho = \rho_0 \qquad 0 < r < R$$

within a medium with permittivity ϵ. The sphere is grounded at $r = R$ so that the scalar electric potential at $r = R$ is zero, $\Phi(r = R) = 0$. There is no point charge at $r = 0$ so that $E_r(r = 0)$ must be finite.

A

Question: What is the EQS electric field $\overline{E}(r)$ for $0 < r < R$?

Solution:

$$\nabla \bullet \overline{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \frac{\rho_o}{\epsilon} \Rightarrow \frac{d}{dr}(r^2 E_r) = \frac{\rho_o r^2}{\epsilon}$$

$$r^2 E_r = \frac{\rho_0 r^3}{3\epsilon} + C_1 \Rightarrow E_r = \frac{\rho_o r}{3\epsilon} + \frac{C_1}{r^2}$$

$$E_r(r = 0) \text{ is finite} \Rightarrow C_1 = 0$$

$$E_r(r) = \frac{\rho_o r}{3\epsilon} \qquad 0 < r < R$$

B

Question: What is the scalar electric potential $\Phi(r)$ where $\overline{E}(r) = -\nabla\Phi(r)$?
Solution:

$$E_r = -\frac{d\Phi}{dr} = \frac{\rho_o r}{3\epsilon} \Rightarrow \Phi = -\frac{\rho_o r^2}{6\epsilon} + C_2$$

$$\Phi(r = R) = 0 = -\frac{\rho_o R^2}{6\epsilon} + C_2 \Rightarrow C_2 = \frac{\rho_o R^2}{6\epsilon}$$

$$\Phi(r) = -\frac{\rho_o}{6\epsilon}(r^2 - R^2)$$

C

Question: What is the free surface charge density $\sigma_s(r = R)$ on the inside surface of the sphere at $r = R$?
Solution:

$$\sigma_s(r = R) = -\epsilon E_r(r = R) = -\frac{\not{\epsilon}\rho_o R}{6\not{\epsilon}} = -\frac{\rho_0 R}{3}$$

Problem 2

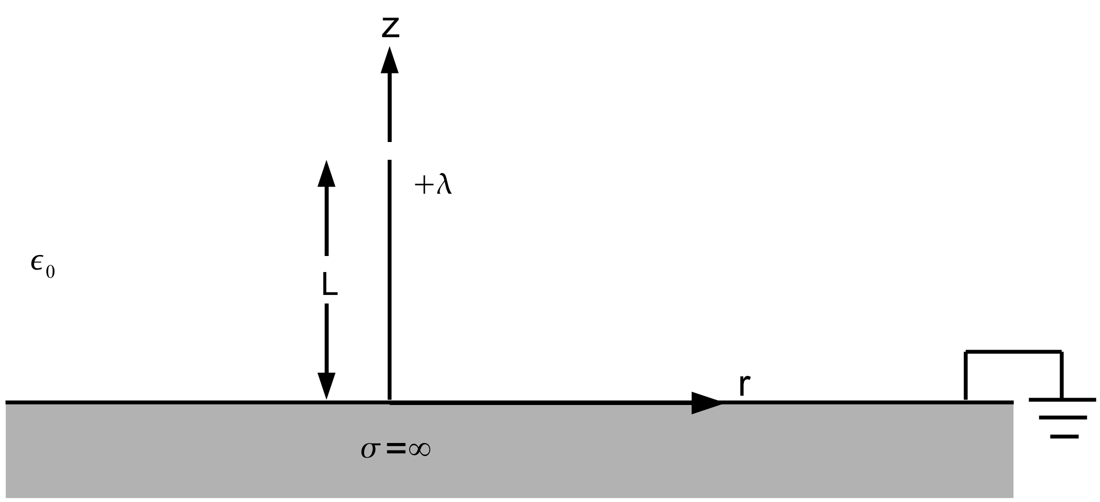

Figure 2: A diagram of a uniform line charge of height L standing perpendicular to the ground plane (Image by MIT OpenCourseWare).

A uniform line charge λ coulombs/meter of height L stands perpendicularly on a perfectly conducting ground plane of infinite extent in free space with dielectric permittivity ϵ_0.

A

Question: Find the electric field at the ground plane surface $\overline{E}(r, z = 0_+)$ where r is the cylindrical radial coordinate shown above. See integrals in hint below.
Solution:

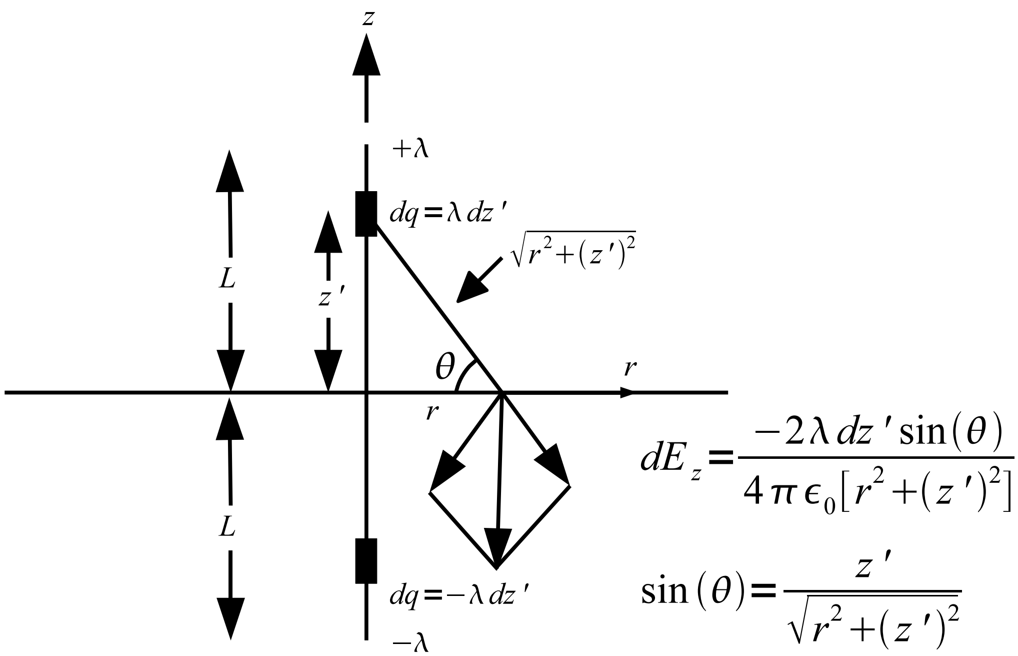

Figure 3: A diagram of the line charge $+\lambda$ in Figure 2 and its image $-\lambda$ and the resulting net $-z$-directed electric field at $z = 0$. (Image by MIT OpenCourseWare).

$$dE_z = -\frac{\lambda dz' z'}{2\pi\epsilon_o[r^2 + (z')^2]^{3/2}}$$

$$E_z(z = 0_+) = -\frac{\lambda}{2\pi\epsilon_o} \int_{z'=0}^{L} \frac{z' dz'}{[r^2 + (z')^2]^{3/2}}$$

$$u = r^2 + (z')^2$$

$$du = 2z' dz'$$

$$E_z(r, z = 0_+) = -\frac{\lambda}{2\pi\epsilon_o} \int_{u=r^2}^{r^2+L^2} \frac{du}{2u^{3/2}} = +\frac{\lambda}{2\pi\epsilon_o u^{1/2}}\Big|_{u=r^2}^{r^2+L^2} = +\frac{\lambda}{2\pi\epsilon_o}\Big[\frac{1}{\sqrt{r^2+L^2}} - \frac{1}{r}\Big]$$

B

Question: Find the surface charge density on the ground plane surface, $\sigma_s(r, z = 0_+)$.
Solution:

$$\sigma_s(r, z = 0_+) = \epsilon_o E_z = +\frac{\lambda}{2\pi}\Big[\frac{1}{\sqrt{r^2+L^2}} - \frac{1}{r}\Big]$$

C

Question: Prove that the total charge $q_t(z=0_+)$ on the ground plane is $-\lambda L$.
Solution:

$$\begin{aligned}
q_t(z=0_+) &= \int_{r=0}^{\infty} \sigma_s(r, z=0_+) 2\pi r dr \\
&= +\int_{r=0}^{\infty} \frac{\cancel{2\pi}\lambda}{\cancel{2\pi}} \Big[\frac{r}{\sqrt{r^2+L^2}} - 1 \Big] dr \\
&= +\lambda[\sqrt{r^2+L^2} - r]|_{r=0}^{\infty} \\
&= +\lambda[(r-r)|_{r\to\infty} - L] \\
&= -\lambda L
\end{aligned}$$

Hint for parts (a) and (c): one or more of the following indefinite integrals may be useful:

i $\int \frac{x dx}{[x^2+L^2]^{1/2}} = \sqrt{x^2+L^2}$

ii $\int \frac{dx}{[x^2+L^2]^{1/2}} = \ln[x + \sqrt{x^2+L^2}]$

iii $\int \frac{dx}{[x^2+L^2]^{3/2}} = \frac{x}{L^2[x^2+L^2]^{1/2}}$

iv $\int \frac{x dx}{[x^2+L^2]^{3/2}} = -\frac{1}{[x^2+L^2]^{1/2}}$

Problem 3

An infinite slab in the y and z directions carries a uniform current density $\overline{J} = J_0 \overline{i_z}$ for $-d < x < d$. The current carrying slab has magnetic permeability of free space μ_0 and is surrounded by free space for $x > d$ and $x < -d$. There are no surface currents on the $x = \pm d$ surfaces, $\overline{K}(x=d) = \overline{K}(x=-d) = 0$ and the magnetic field only depends on the x coordinate.

A

Question: Find the magnetic field $\overline{H}(x)$ everywhere and plot versus x.
Solution:

$$\nabla \times \overline{H} = \overline{J} = J_o \overline{i_z} \Rightarrow \frac{\partial H_y}{\partial x} - \cancelto{0}{\frac{\partial H_x}{\partial y}} = J_o \qquad -d \le x \le d$$

$$H_y(x) = J_o x + C \qquad -d \le x \le d \quad , \quad C = 0 \text{ by symmetry}$$

$$H_y(x = d_-) = H_y(x = d_+) = J_o d \quad , \quad H_y(x \ge d) = J_o d$$

$$H_y(x = -d_-) = H_y(x = -d_+) = -J_o d \quad , \quad H_y(x \le -d) = -J_o d$$

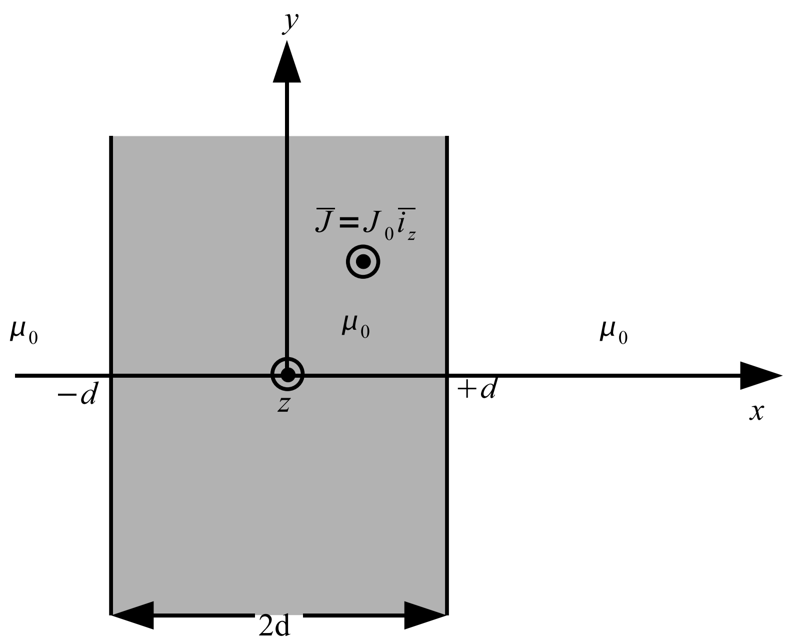

Figure 4: A diagram of an infinite slab in the y and z directions with width $2d$ carrying a uniform current density $J_0\vec{i}_z$. (Image by MIT OpenCourseWare).

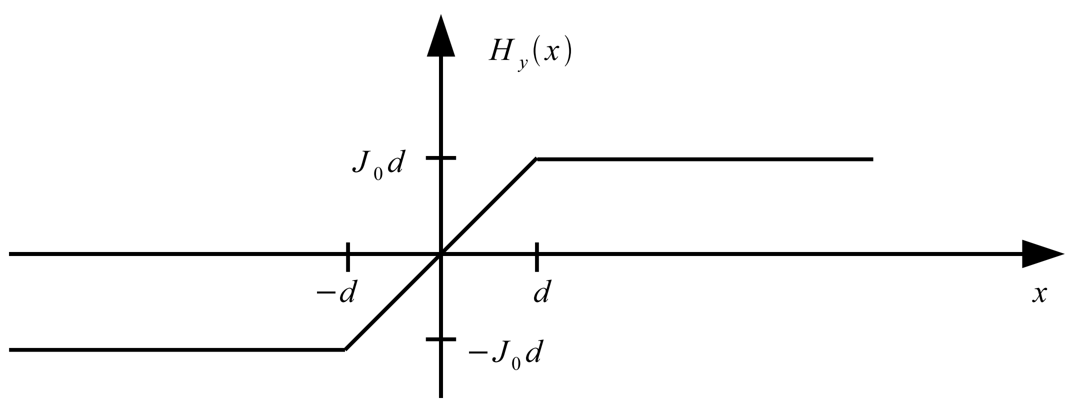

Figure 5: A graph showing the magnetic field versus the x coordinate (Image by MIT OpenCourseWare).

B

Question:

A small cylindrical hole of radius a and of infinite extent in the z direction is drilled into the current carrying slab of part (a) and is centered within the slab at the origin. The magnetic permeability of all regions is μ_0. Within the hole for $r < a$ the current density is zero, $\overline{J} = 0$. What is the total magnetic field $\overline{H}$ in the hole?

Hint 1: Use superposition replacing the cylindrical hole by two oppositely directed currents.

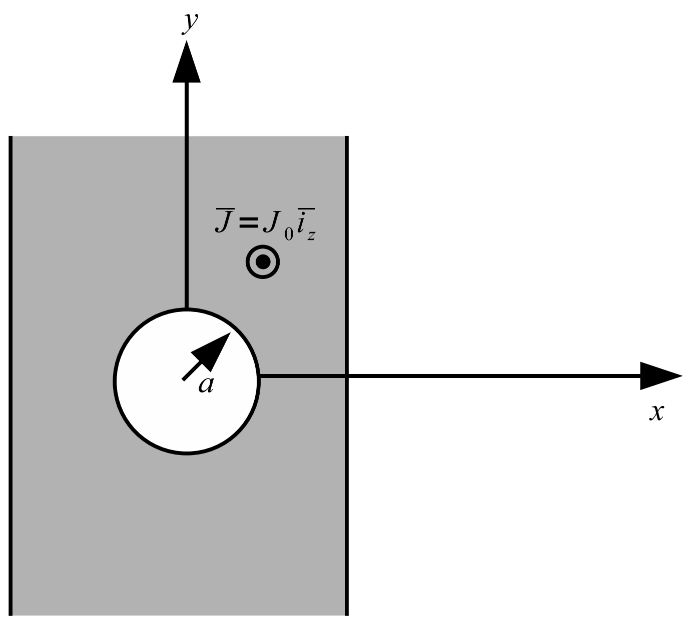

Figure 6: A diagram of an infinite slab in the y and z directions with a cylindrical hole of radius a at the origin with infinite extent in the z direction (Image by MIT OpenCourseWare).

Hint 2: $r\overline{i_\phi} = r(-\sin\phi\overline{i_x} + \cos\phi\overline{i_y}) = (-y\overline{i_x} + x\overline{i_y})$ **where** $r = \sqrt{x^2 + y^2}$

Solution: Step 1: Put current density $J_o\overline{i_z}$ in hole with $J_o\overline{i_z}$ outside hole. Then magnetic field is the same as part (a) with $\overline{J} = 0$ outside

Step 2: Put current density $-J_o\overline{i_z}$ in hole with $\overline{J} = 0$ outside hole. Thus net current in hole is zero and net current outside hole is $J_o\overline{i_z}$.

For step 1: $\overline{H_1} = J_o x\overline{i_y}$ in hole

For step 2: $\nabla \times \overline{H_2} = -J_o\overline{i_z} = \overline{i_z}\frac{1}{r}\left[\frac{\partial(rH_{\phi 2})}{\partial r} - \cancelto{0}{\frac{\partial H_{r2}}{\partial \phi}}\right]$

$$\frac{1}{\cancel{r}}\frac{\partial(rH_{\phi 2})}{\partial r} = -J_o r$$

$$rH_{\phi 2} = -\frac{J_o r^2}{2} + C$$

$$H_{\phi 2} = -\frac{J_o r}{2} + \frac{C}{r}$$

$$H_{\phi 2}(r = 0) = \text{ finite } \Rightarrow C = 0$$

$$H_{\phi 2} = -\frac{J_o r}{2}$$

$$\overline{H_2} = -\frac{J_o r}{2}\overline{i_\phi} = -\frac{J_o}{2}(-y\overline{i_x} + x\overline{i_y})$$

$$\begin{aligned}\overline{H_T} &= \overline{H_1} + \overline{H_2} \qquad \text{(in hole)}\\ &= J_o x\overline{i_y} - \frac{J_o}{2}(-y\overline{i_x} + x\overline{i_y})\\ &= \frac{J_o}{2}(x\overline{i_y} + y\overline{i_x})\end{aligned}$$

C

Question: Verify that your solution of part (b) satisfies the MQS Ampere's law within the hole where $\overline{J} = 0$.
Solution:

$$\nabla \times \overline{H} = \overline{i_z}\Big[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\Big] = \frac{J_o}{2}[1-1]\overline{i_z} = 0$$

Problem 4

A resistor is formed in the shape of a circular cylindrical half-shell of inner radius b and outer radius a and is composed of two materials with ohmic conductivites and permittivities (σ_1, ϵ_1) for $0 < \phi < \frac{\pi}{2}$ and (σ_2, ϵ_2) for $\frac{\pi}{2} < \phi < \pi$. A DC voltage V_0 is applied to the electrode at $\phi = 0$ while the electrode at $\phi = \pi$ is grounded. The EQS scalar potential is thus imposed as $\Phi(\phi = 0) = V_0, \Phi(\phi = \pi) = 0$. The cylindrical system has a depth d.

A

Question: The solution for the EQS scalar potential is each conducting material can be written in the form

$$\Phi_1 = A_1\phi + B_1 \quad 0 < \phi < \frac{\pi}{2}$$

$$\Phi_2 = A_2\phi + B_2 \quad \frac{\pi}{2} < \phi < \pi$$

In the dc steady state what are the boundary conditions that allow calculation of A_1, A_2, B_1, and B_2? Find A_1, A_2, B_1, and B_2.
Solution: Boundary Conditions

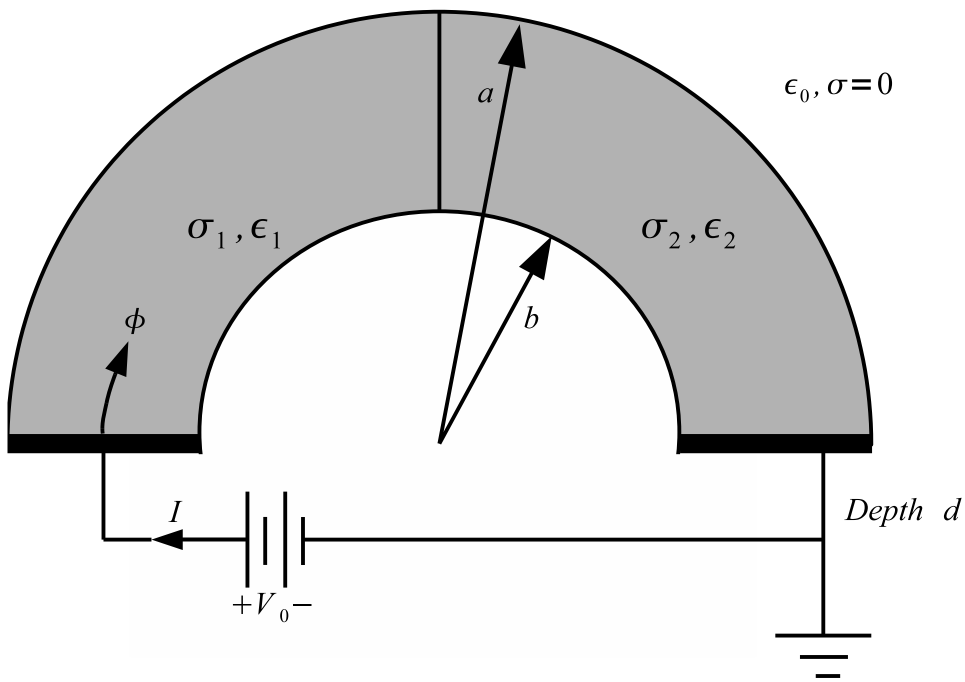

Figure 7: A diagram of a semi-circular shaped resistor formed from two different materials (Image by MIT OpenCourseWare).

$$\Phi_1(\phi = 0) = V_o = B_1$$
$$\Phi_2(\phi = \pi) = 0 = A_2\pi + B_2$$
$$\Phi_1\left(\phi = \frac{\pi}{2}\right) = \Phi_2\left(\phi = \frac{\pi}{2}\right) \Rightarrow A_1\frac{\pi}{2} + B_1 = A_2\frac{\pi}{2} + B_2$$
$$\sigma_1 E_{\phi 1}\left(\phi = \frac{\pi}{2}\right) = \sigma_2 E_{\phi 2}\left(\phi = \frac{\pi}{2}\right), \qquad E_{\phi 1} = -\frac{1}{r}\frac{\partial \Phi_1}{\partial \phi} = -\frac{A_1}{r}$$
$$\sigma_1 A_1 = \sigma_2 A_2 \qquad E_{\phi 2} = -\frac{1}{r}\frac{\partial \Phi_2}{\partial \phi} = -\frac{A_2}{r}$$

$$B_1 = V_0$$
$$B_2 = -A_2\pi$$
$$A_1\frac{\pi}{2} + B_1 = A_2\frac{\pi}{2} - A_2\pi = -A_2\frac{\pi}{2} \Rightarrow A_1\frac{\pi}{2} + V_0 = -A_2\frac{\pi}{2}$$
$$\sigma_1 A_1 = \sigma_2 A_2 \Rightarrow A_2 = \frac{\sigma_1}{\sigma_2}A_1$$
$$\frac{\pi}{2}[A_1 + A_2] = -V_0$$
$$A_1\left[1 + \frac{\sigma_1}{\sigma_2}\right] = -\frac{2V_0}{\pi}$$
$$A_1 = -\frac{2\sigma_2 V_0}{\pi[\sigma_1 + \sigma_2]}$$
$$A_2 = -\frac{2\sigma_1 V_0}{\pi[\sigma_1 + \sigma_2]}$$
$$B_2 = \frac{2\sigma_1 V_0}{[\sigma_1 + \sigma_2]},\ B_1 = V_0$$

B

<u>Question:</u> What is the electric field in each region of the resistor?
<u>Solution:</u>

$$\overline{E_1} = -\frac{A_1}{r}\overline{i_\phi} = \frac{2\sigma_2 V_0}{r\pi(\sigma_1 + \sigma_2)}\overline{i_\phi}$$
$$\overline{E_2} = -\frac{A_2}{r}\overline{i_\phi} = \frac{2\sigma_1 V_0}{r\pi(\sigma_1 + \sigma_2)}\overline{i_\phi}$$

C

<u>Question:</u> What are the free surface charge densities on the interfaces at $\phi = 0,\ \phi = \frac{\pi}{2},$ and $\phi = \pi$?
<u>Solution:</u>

$$\sigma_s(r, \phi = 0) = \epsilon_1 E_{\phi 1}(\phi = 0) = \frac{\epsilon_1 V_0}{r}\frac{2\sigma_2}{\pi(\sigma_1 + \sigma_2)}$$
$$\sigma_s(r, \phi = \pi) = -\epsilon_2 E_{\phi 2}(\phi = \pi) = -\frac{\epsilon_2 V_0}{r}\frac{2\sigma_1}{\pi(\sigma_1 + \sigma_2)}$$
$$\begin{aligned}\sigma_s\left(r, \phi = \frac{\pi}{2}\right) &= \epsilon_2 E_{\phi 2}\left(\phi = \frac{\pi}{2}+\right) - \epsilon_1 E_{\phi 1}\left(\phi = \frac{\pi}{2}-\right) \\ &= -\frac{[\epsilon_2 A_2 - \epsilon_1 A_1]}{r} \\ &= \frac{-2V_0}{r\pi(\sigma_1 + \sigma_2)}[-\epsilon_2\sigma_1 + \epsilon_1\sigma_2] \\ &= \frac{2V_0}{r\pi(\sigma_1 + \sigma_2)}[\epsilon_2\sigma_1 - \epsilon_1\sigma_2]\end{aligned}$$

D

Question: What is the DC terminal current I that flows from the battery?
Solution:

$$I = \int_{r=b}^{a} J_\phi d dr = \int_{r=b}^{a} \sigma_1 E_{1\phi} d dr = \int_{r=b}^{a} \frac{\sigma_1 2\sigma_2 V_0 d}{\pi r(\sigma_1 + \sigma_2)} dr$$
$$= \frac{2\sigma_1\sigma_2 d V_0}{\pi(\sigma_1 + \sigma_2)} \ln\frac{a}{b}$$

E

Question: What is the resistance between the electrodes at $\phi = 0$ and $\phi = \pi$?
Solution:

$$R = \frac{V_0}{I} = \frac{\pi(\sigma_1 + \sigma_2)}{2\sigma_1\sigma_2 d \ln\frac{a}{b}}$$