6.641 Electromagnetic Fields, Forces, and Motion
Spring 2005

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2005
Problem Set 3 - Solutions	
Prof. Markus Zahn	MIT OpenCourseWare

Problem 3.1

A

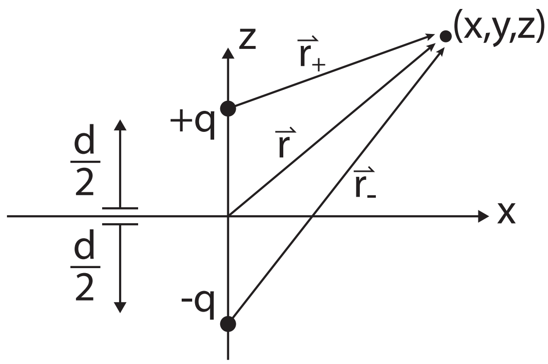

Figure 1: Addition of potential contributions from 2 point charges. (Image by MIT OpenCourseWare.)

We can simply add the potential contributions of each point charge:

$$\Phi = \frac{q}{4\pi\varepsilon_0 r_+} - \frac{q}{4\pi\varepsilon_0 r_-}$$

$$r_+ = \sqrt{x^2 + y^2 + (z - \frac{d}{2})^2}$$

$$r_- = \sqrt{x^2 + y^2 + (z + \frac{d}{2})^2}$$

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2 + y^2 + (z - \frac{d}{2})^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + \frac{d}{2})^2}}\right]$$

B

$p = qd$. We must make some approximations. As $r \to \infty$, $\vec{r}_+$, $\vec{r}_-$, and $\vec{r}$ become nearly parallel. Thus,

$$r_+ \approx r - a = r - \frac{d}{2}\cos\theta$$

$$r_+ \approx r(1 - \frac{d}{2r}\cos\theta)$$

Similarly,

$$r_- \approx r(1 + \frac{d}{2r}\cos\theta)$$

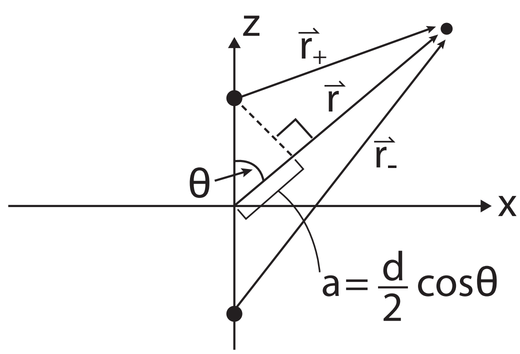

Figure 2: Differences in lengths between $\vec{r}_+$, $\vec{r}_-$, and $\vec{r}$ (Image by MIT OpenCourseWare.)

By part (a): $\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right]$. If $|x| \ll 1$, then $\frac{1}{1+x} \approx 1 - x$

$$|\frac{d}{2r}\cos\theta| \ll 1$$

so

$$\frac{1}{r_+} \approx \frac{1}{r}\frac{1}{1 - \frac{d}{2r}\cos\theta} \approx \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right)$$

$$\frac{1}{r_-} \approx \frac{1}{r}\frac{1}{1 + \frac{d}{2r}\cos\theta} \approx \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)$$

$$\Rightarrow \frac{1}{r_+} - \frac{1}{r_-} \approx \frac{1}{r}\frac{d}{r}\cos\theta = \frac{d}{r^2}\cos\theta$$

$$\Phi \approx \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

C

$$\overrightarrow{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial r}\hat{i}_r - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{i}_\theta - \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{i}_\phi$$

$$\frac{\partial\Phi}{\partial r} = -\frac{p\cos\theta}{2\pi\varepsilon_0 r^3}; \frac{\partial\Phi}{\partial\theta} = -\frac{p\sin\theta}{4\pi\varepsilon_0 r^2}$$

$$\frac{\partial\Phi}{\partial\phi} = 0$$

$$\overrightarrow{E} = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}\hat{i}_r + \frac{1}{r}\frac{p\sin\theta}{4\pi\varepsilon_0 r^2}\hat{i}_\theta$$

$$\overrightarrow{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\hat{i}_r + \sin\theta\hat{i}_\theta\right]$$

D

$$\frac{dr}{rd\theta} = \frac{E_r}{E_\theta} = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta$$

$$\frac{1}{r}dr = 2\cot\theta d\theta$$

$$\int \frac{1}{r}dr = \int 2\cot\theta d\theta$$

$$\ln r = 2\ln(\sin\theta) + k$$

$$r = C\sin^2\theta; \qquad \text{when } \theta = \frac{\pi}{2}, r = C = r_0$$

Thus,

$$C = r_0, \frac{r}{r_0} = \sin^2\theta$$

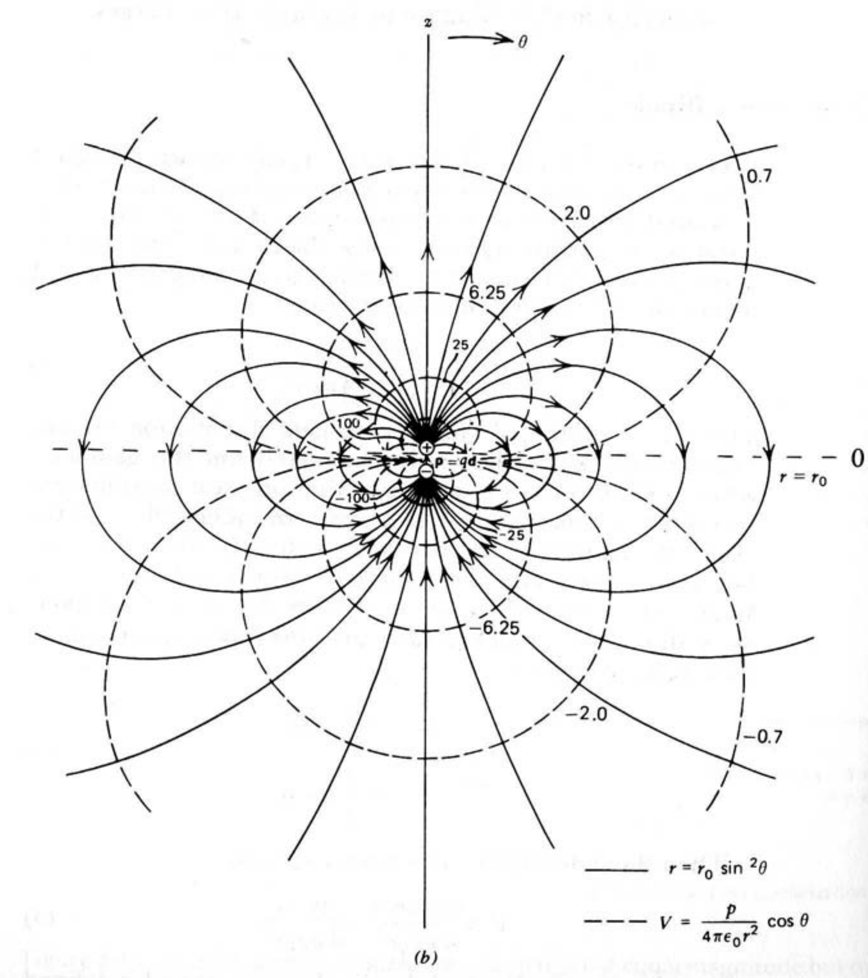

Figure 3.2b) Used with permission.
Zahn, Markus. In *Electromagnetic Field Theory: A Problem Solving Approach*, 1987.

Plots of Equipotential and Field Lines

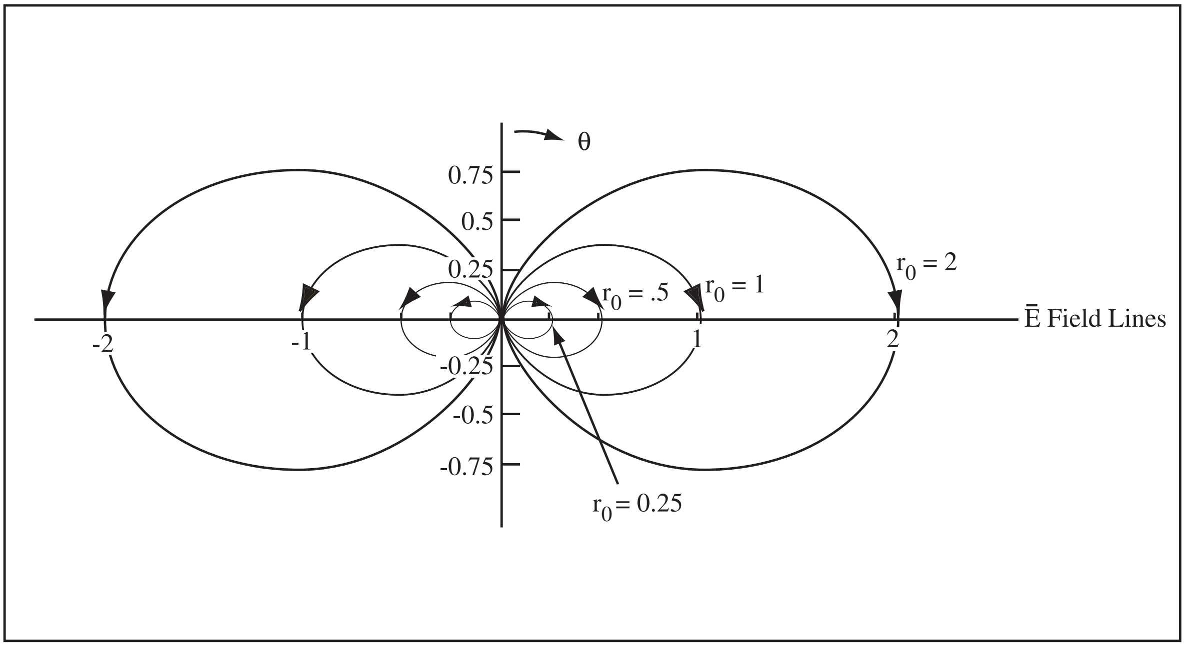

Figure 4: Polar plot of $r_0 \sin^2\theta$ for $0 \leq \theta \leq \pi$ and for $r_0 = 0.25$, 0.5, 1, and 2 meters with $\frac{4\pi\varepsilon_0}{p} = 100$ volt^{-1}-m^{-2} (Image by MIT OpenCourseWare.)

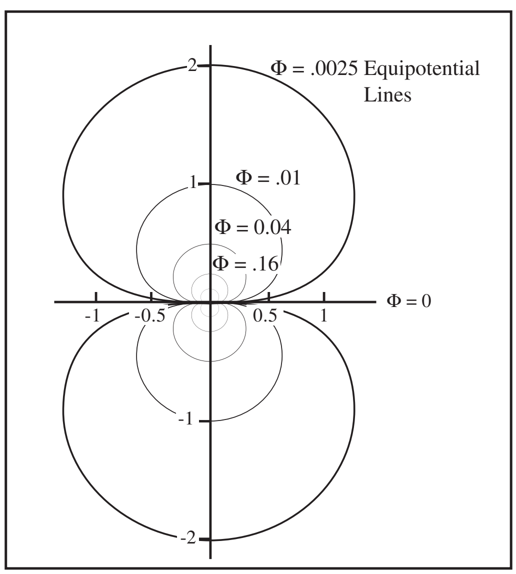

Figure 5: Polar plot of equipotential lines $\Phi = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$ for $0 \leq \theta \leq \pi$, $\Phi = 0, \pm 0.0025, \pm 0.01, \pm 0.04, \pm 0.16,$ and ± 0.64 volts with $\frac{4\pi\varepsilon_0}{p} = 100$ volt^{-1}-m^{-2} (Image by MIT OpenCourseWare.)

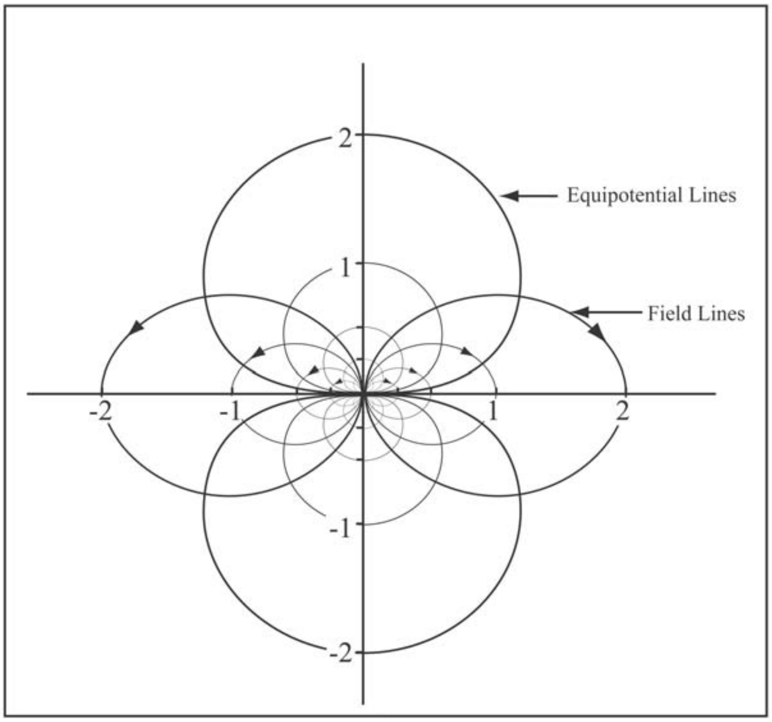

Figure 6: The superposition of the previous two plots of perpendicular equipotential and field lines (Image by MIT OpenCourseWare.)

Problem 3.2

A

We can think of the bird as a perfectly conducting small sphere when it lands on the uninsulated wire, it must become the same potential as the wire. This forces it to acquire a charge. When it flies away, the charge stays with it because air is a poor conductor.

B, C

For B and C, use the method of images. We can use superposition to get total potential.

$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{[(x-Ut)^2+(y-h)^2+z^2]^{\frac{1}{2}}} - \frac{1}{[(x-Ut)^2+(y+h)^2+z^2]^{\frac{1}{2}}}\right]$$

where q is the charged bird modeled as a point charge.

D

By boundary condition found using Gauss' Law

$$\hat{n}\cdot(\varepsilon_a\overrightarrow{E}^a - \varepsilon_b\overrightarrow{E}^b) = \sigma_s$$

at $y = 0$ ground plane boundary

$$\overrightarrow{E}^b = 0$$

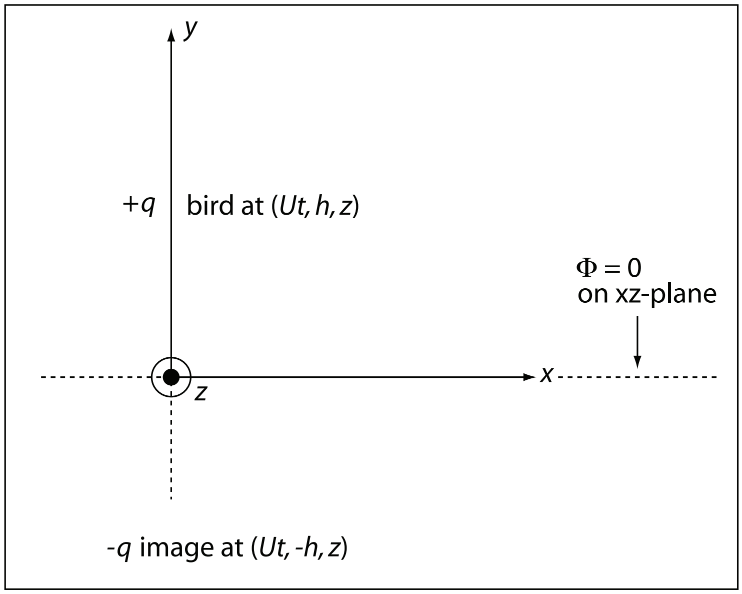

Figure 7: Figure for 3.2 B, C. Method of Images for charged bird taken as a point charge (Image by MIT OpenCourseWare.)

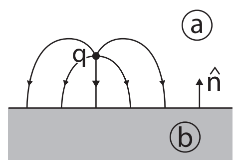

Figure 8: Figure for 3.2 D. Field lines from point charge above a perfectly conducting ground plane (Image by MIT OpenCourseWare.)

Because we can consider ground plane to be a perfect conductor, $\hat{n} \cdot \overrightarrow{E}^a = \frac{\sigma_s}{\varepsilon_0}$.

$$(\hat{i}_y) \cdot (\overrightarrow{E}(x, y = 0^+, z)) = \frac{\sigma_s}{\varepsilon_0} \text{ implies we only care about } y \text{ components of } \overrightarrow{E}$$

$$E_y(x, y = 0^+, z) = \frac{\sigma_s}{\varepsilon_0} \tag{1}$$

$$E_y = -\frac{\partial}{\partial y}\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{(y-h)}{[(x-Ut)^2 + (y-h)^2 + z^2]^{\frac{3}{2}}} - \frac{(y+h)}{[(x-Ut)^2 + (y+h)^2 + z^2]^{\frac{3}{2}}}\right]$$

Evaluate at $y = 0$ and substitute into (1) above:

$$E_y(x, y = 0, z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{-2h}{[(x-Ut)^2 + h^2 + z^2]^{\frac{3}{2}}}\right]$$

So $\sigma_s = \varepsilon_0 E_y(x, y = 0, z)$

$$\sigma_s = \frac{-qh}{2\pi\,[(x-Ut)^2 + h^2 + z^2]^{\frac{3}{2}}}$$

E

$$Q = \int_0^w \int_0^l \frac{-qh}{2\pi\left[(x-Ut)^2+h^2+z^2\right]^{\frac{3}{2}}}dxdz$$

For w very small, σ_s does not change significantly from $z = 0$ to $z = w$, so integral in z becomes just multiplication at $z = 0$.

$$Q = \int_0^l \frac{-qhw}{2\pi\left[(x-Ut)^2+h^2\right]}dx$$

Let $x' = x - Ut \Rightarrow dx' = dx$ So:

$$Q = \int_{-Ut}^{l-Ut} \frac{-qhw}{2\pi\left[((x')^2+h^2\right]^{\frac{3}{2}}}dx'$$

$$Q = -\frac{qw}{2\pi h}\left[\frac{l-Ut}{\sqrt{(l-Ut)^2+h^2}}+\frac{Ut}{\sqrt{(Ut)^2+h^2}}\right]$$

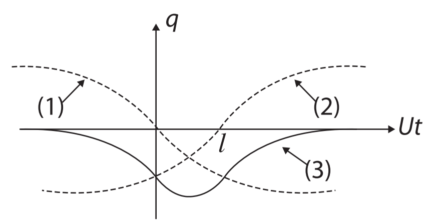

Figure 9: Representative shape $q = Q$ versus Ut (Image by MIT OpenCourseWare.)

F

$$i = \frac{dQ}{dt} = \frac{-qw}{2\pi h}\left[\frac{-Uh^2}{\left[(l-Ut)^2+h^2\right]^{\frac{3}{2}}}+\frac{Uh^2}{\left[(Ut)^2+h^2\right]^{\frac{3}{2}}}\right]$$

$$V = -iR = \frac{qwR}{2\pi h}\left[\frac{-Uh^2}{\left[(l-Ut)^2+h^2\right]^{\frac{3}{2}}}+\frac{Uh^2}{\left[(Ut)^2+h^2\right]^{\frac{3}{2}}}\right]$$

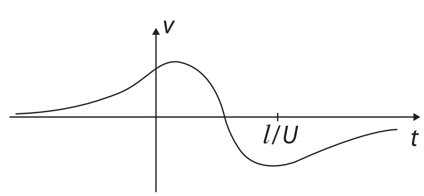

Figure 10: Voltage V versus time across small electrode resistance R (Image by MIT OpenCourseWare.)

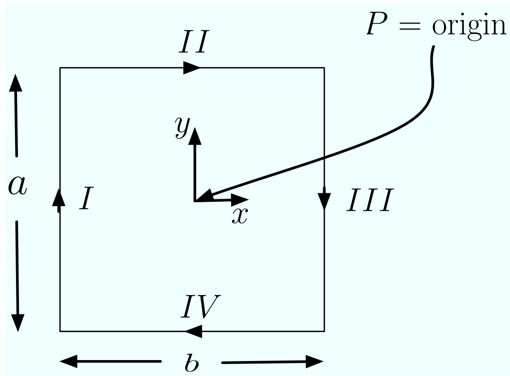

Figure 11: Magnetic field from rectangular line current (Image by MIT OpenCourseWare.)

Problem 3.3

A

$$\overrightarrow{H} = \frac{1}{4\pi}\int \frac{J(\overrightarrow{r}') \times \hat{i}_{r'r}}{|\overrightarrow{r} - \overrightarrow{r}'|^2} dv'$$

$$\overrightarrow{H} = \frac{1}{4\pi}\int_{\substack{y=-\frac{a}{2}\\ z=0\\ x=-\frac{b}{2}}}^{y=\frac{a}{2}} \frac{(I\hat{i}_y) \times \left(\frac{b}{2}\hat{i}_x - y\hat{i}_y\right) dy}{\underbrace{\left(\left(\frac{b}{2}\right)^2 + y^2\right)}_{|\overrightarrow{r}-\overrightarrow{r}'|^2}\underbrace{\left(\left(\frac{b}{2}\right)^2 + y^2\right)^{\frac{1}{2}}}_{\text{normalization for } \hat{i}_{r'r}}} + \int_{\substack{x=-\frac{b}{2}\\ z=0\\ y=\frac{a}{2}}}^{x=\frac{b}{2}} \frac{(I\hat{i}_x) \times \left(-x\hat{i}_x - \frac{a}{2}\hat{i}_y\right) dx}{\left(\left(\frac{a}{2}\right)^2 + x^2\right)\left(\left(\frac{a}{2}\right)^2 + x^2\right)^{\frac{1}{2}}} +$$

$$\int_{\substack{y=-\frac{a}{2}\\ x=\frac{b}{2}\\ z=0}}^{y=\frac{a}{2}} \frac{(-I\hat{i}_y) \times \left(-\frac{b}{2}\hat{i}_x - y\hat{i}_y\right) dy}{\left(\left(\frac{b}{2}\right)^2 + x^2\right)\left(\left(\frac{b}{2}\right)^2 + x^2\right)^{\frac{1}{2}}} + \int_{\substack{x=-\frac{b}{2}\\ y=-\frac{a}{2}\\ z=0}}^{x=\frac{b}{2}} \frac{(-I\hat{i}_x) \times \left(-x\hat{i}_x + \frac{a}{2}\hat{i}_y\right) dx}{\left(\left(\frac{a}{2}\right)^2 + x^2\right)\left(\left(\frac{a}{2}\right)^2 + x^2\right)^{\frac{1}{2}}}$$

$$= \frac{1}{4\pi}\left[2\int_{x=-\frac{b}{2}}^{x=\frac{b}{2}} \frac{a}{2}\frac{I(-\hat{i}_z)dx}{\left(\left(\frac{a}{2}\right)^2 + x^2\right)^{\frac{3}{2}}} + 2\int_{y=-\frac{a}{2}}^{y=\frac{a}{2}} \frac{b}{2}\frac{I(-\hat{i}_z)dy}{\left(\left(\frac{b}{2}\right)^2 + x^2\right)^{\frac{3}{2}}}\right]$$

$$= -\frac{I\hat{i}_z}{4\pi}\left[\left.\frac{ax}{\left(\frac{a}{2}\right)^2\left(\left(\frac{a}{2}\right)^2 + x^2\right)^{\frac{1}{2}}}\right|_{-\frac{b}{2}}^{\frac{b}{2}} + \left.\frac{by}{\left(\frac{b}{2}\right)^2\left(\left(\frac{b}{2}\right)^2 + y^2\right)^{\frac{1}{2}}}\right|_{-\frac{a}{2}}^{\frac{a}{2}}\right]$$

$$\overrightarrow{H} = -\frac{I\hat{i}_z}{\cancel{4}\pi}\left[\frac{\cancel{4}ab}{a^2\left(\left(-\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right)^{\frac{1}{2}}} + \frac{\cancel{4}ab}{b^2\left(\left(\frac{b}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right)^{\frac{1}{2}}}\right]$$

$$\overrightarrow{H} = \frac{-2I\left(a^2 + b^2\right)\hat{i}_z}{\pi ab(a^2 + b^2)^{\frac{1}{2}}}$$

$$\overrightarrow{H} = \frac{-2I(a^2+b^2)^{\frac{1}{2}}}{\pi ab}\hat{i}_z$$

B

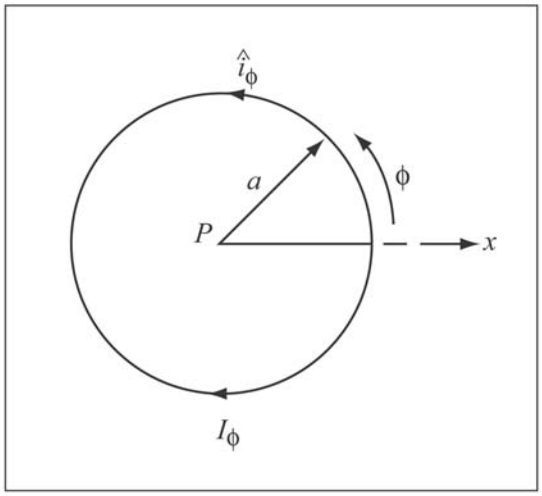

Figure 12: Line current in circular coil (Image by MIT OpenCourseWare.)

$$\overrightarrow{I} = -I\hat{i}_\phi$$

$$\begin{aligned}\overrightarrow{H} &= \frac{1}{4\pi}\int_0^{\pi}\frac{\left(-I\hat{i}_\phi\right)\times(-\hat{i}_r)ad\phi}{a^2}\\ &= \frac{1}{4\pi}\int_0^{2\pi}\frac{-\hat{i}_z Iad\phi}{a^2}\end{aligned}$$

$$\overrightarrow{H} = -\frac{I}{2a}\hat{i}_z$$

C

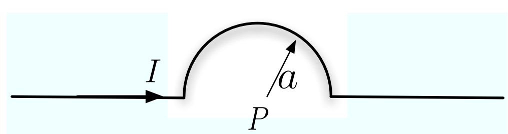

Figure 13: Line current with semi-circular bump (Image by MIT OpenCourseWare.)

$$\overrightarrow{H} = \frac{1}{4\pi}\int_0^{\pi}\frac{(-I\hat{i}_\phi)\times(-\hat{i}_r)ad\phi}{a^2}$$

$$\overrightarrow{H} = -\frac{I}{4a}\hat{i}_z$$

D

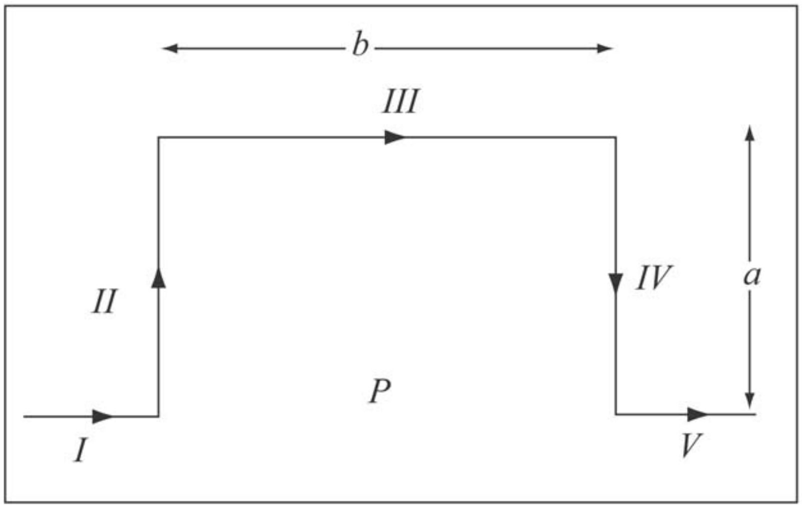

Figure 14: Line current with rectangular bump (Image by MIT OpenCourseWare.)

As in part (c), contributions from segments I and V are zero (see Fig. 14). Segments II, III, and IV are just like part (a), except integrals in y are from 0 to a and only one integral in x and $(\frac{a}{2}) \rightarrow a$.

$$\overrightarrow{H} = \frac{-I\left(a^2 + \left(\frac{b}{2}\right)^2\right)^{\frac{1}{2}}}{\pi ab}\hat{i}_z$$

Problem 3.4

A

$$\overrightarrow{H} = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{(K_0\hat{i}_\phi)\times\hat{i}_{r'r}R^2\sin\theta d\phi d\theta}{|\overrightarrow{r}-\overrightarrow{r}'|^2} \qquad (\overrightarrow{r}=0)$$

$$\hat{i}_{r'r} = \frac{\overrightarrow{r}-\overrightarrow{r}'}{|\overrightarrow{r}-\overrightarrow{r}'|} = -\frac{\overrightarrow{r}'}{|\overrightarrow{r}'|} = -\hat{i}'_r$$

$$\overrightarrow{H} = \frac{K_0}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{(\hat{i}_\phi)\times(-\hat{i}_r)\cancel{R^2}}{\cancel{R^2}}\sin\theta d\phi d\theta$$

$$= -\frac{K_0}{4\pi}\int_0^{\pi}\int_0^{2\pi}(\sin\theta)(\cos\theta\cos\phi\hat{i}_x + \cos\theta\sin\phi\hat{i}_y - \sin\theta\hat{i}_z)d\phi d\theta$$

Any term with an odd power of sin or cos in ϕ integrates to 0 in ϕ because integral is over one period. $\overrightarrow{H} = \hat{i}_z\frac{2\pi K_0}{4\pi}\int_0^{\pi}\sin^2\theta d\theta = \frac{K_0\pi}{4}\hat{i}_z = \overrightarrow{H}$

B

This requires us to integrate an infinite number of infinitesimal current shells of the type in (a) from $r = R_1$ to R_2.

$$\overrightarrow{H} = \int_{R_1}^{R_2} \frac{\overbrace{(J_0 dr)}^{K_0}\pi}{4}\hat{i}_z dr \Rightarrow \overrightarrow{H} = \frac{J_0\pi}{4}(R_2 - R_1)\hat{i}_z$$

Problem 3.5

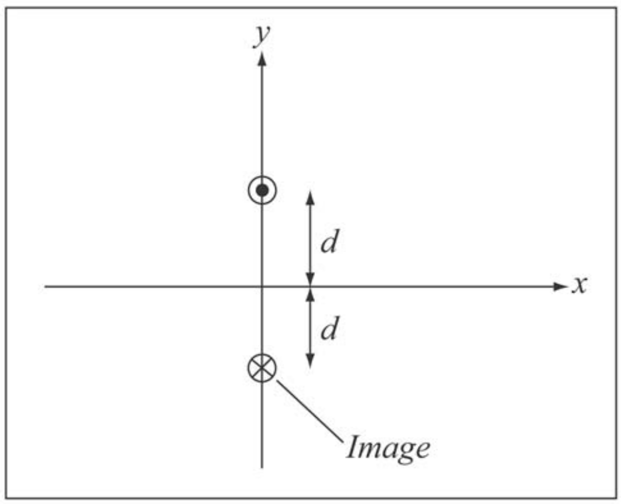

Figure 15: Line current above perfectly conducting plane with image current (Image by MIT OpenCourseWare.)

A

$$\overrightarrow{A} = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} \frac{Idz'\hat{i}_z}{\sqrt{x^2 + (y-d)^2 + (z-z')^2}} - \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} \frac{Idz'\hat{i}_z}{\sqrt{x^2 + (y+d)^2 + (z-z')^2}}$$

Let $\xi \equiv z' - z \Rightarrow d\xi = dz'$. Both integrands are even functions in ξ.

$$\begin{aligned}
\overrightarrow{A} &= \hat{i}_z \frac{\mu_0 I}{2\pi}\int_0^{\infty}\left(\frac{1}{\sqrt{x^2+(y-d)^2+\xi^2}} - \frac{1}{\sqrt{x^2+(y+d)^2+\xi^2}}\right)d\xi \\
&= \hat{i}_z \frac{\mu_0 I}{2\pi}\left[\ln\left\{\frac{\sqrt{x^2+(y-d)^2+\xi^2}}{\sqrt{x^2+(y-d)^2}} + \frac{\xi}{\sqrt{x^2+(y-d)^2}}\right\}\right]_{\xi=0}^{\xi=\infty} \\
&\quad - \left[\hat{i}_z \frac{\mu_0 I}{2\pi}\ln\left\{\frac{\sqrt{x^2+(y+d)^2+\xi^2}}{\sqrt{x^2+(y+d)^2}} + \frac{\xi}{\sqrt{x^2+(y+d)^2}}\right\}\right]_{\xi=0}^{\xi=\infty}
\end{aligned}$$

$$\overrightarrow{A} = \frac{\mu_0 I}{2\pi}\ln\left[\frac{\sqrt{x^2+(y+d)^2}}{\sqrt{x^2+(y-d)^2}}\right]\hat{i}_z$$

B

$$\begin{aligned}\overrightarrow{H} &= \frac{1}{\mu_0}\nabla \times \overrightarrow{A} \\ &= \left[\frac{I(y+d)}{2\pi(x^2+(y+d)^2)} - \frac{I(y-d)}{2\pi(x^2+(y-d)^2)}\right]\hat{i}_x - \left[\frac{Ix}{2\pi(x^2+(y+d)^2)} - \frac{Ix}{2\pi(x^2+(y-d)^2)}\right]\hat{i}_y\end{aligned}$$

C

$$-H_x\big|_{y=0^+} = K_z$$

$$\overrightarrow{K} = -\frac{Id}{\pi(x^2+d^2)}\hat{i}_z$$

D

Force comes from the image current

$$\begin{aligned}\overrightarrow{F} &= (Il\hat{i}_z) \times (\mu_0\overrightarrow{H}(x=0, y=d)) \\ &= \frac{\mu_0 I^2 l}{4\pi d}\hat{i}_y\end{aligned}$$

$$\frac{F}{l} = \frac{\mu_0 I^2}{4\pi d}\hat{i}_y$$