Wednesday, May 4, 2005
Handout #23
Due: never

Problem Set 10

Problem 10.1 (Mod-2 lattices and trellis codes)

(a) Let $\mathcal{C}$ be an (n, k, d) binary linear block code. Show that

$$\Lambda_{\mathcal{C}} = \{\mathbf{x} \in \mathbb{Z}^n \mid \mathbf{x} \equiv \mathbf{c} \bmod 2 \text{ for some } \mathbf{c} \in \mathcal{C}\}$$

is an n-dimensional sublattice of $\mathbb{Z}^n$ (called a "Construction A" or "mod-2" lattice).

(b) Show that if $\mathcal{C}$ has N_d weight-d words, then the mod-2 lattice $\Lambda_{\mathcal{C}}$ has the following geometrical parameters:

$$\begin{aligned}
d^2_{\min}(\Lambda_{\mathcal{C}}) &= \min\{d, 4\}; \\
K_{\min}(\Lambda_{\mathcal{C}}) &= \begin{cases} 2^d N_d, & \text{if } d < 4; \\ 2n, & \text{if } d > 4; \\ 2^d N_d + 2n, & \text{if } d = 4; \end{cases} \\
V(\Lambda_{\mathcal{C}}) &= 2^{n-k}; \\
\gamma_c(\Lambda_{\mathcal{C}}) &= \frac{d^2_{\min}(\Lambda_{\mathcal{C}})}{2^{\eta(\mathcal{C})}},
\end{aligned}$$

where $\eta(\mathcal{C}) = 2(n - k)/n$ is the redundancy of $\mathcal{C}$ in bits per two dimensions.

(c) Show that the mod-2 lattices corresponding to the $(4, 3, 2)$ and $(4, 1, 4)$ binary linear block codes have coding gain $2^{1/2}$ (1.51 dB) (these lattices are in fact versions of D_4). Show that the mod-2 lattice corresponding to the $(8, 4, 4)$ binary linear block code has coding gain 2 (3.01 dB) (this lattice is in fact a version of E_8). Show that no mod-2 lattice has a nominal coding gain more than 4 (6.02 dB).

(d) Let $\mathcal{C}$ be a rate-k/n binary linear convolutional code with free distance d and N_d minimum-weight code sequences per n dimensions. Define the corresponding mod-2 trellis code $\Lambda_{\mathcal{C}}$ to be the set of all integer sequences $\mathbf{x}$ with D-transform $x(D)$ such that $x(D) \equiv c(D) \bmod 2$ for some code sequence $c(D) \in \mathcal{C}$.

(i) Show that an encoder as in Figure 5 of Chapter 14 based on the convolutional code $\mathcal{C}$ and the lattice partition $\mathbb{Z}^n/2\mathbb{Z}^n$ is an encoder for this mod-2 trellis code.

(ii) Show that $\Lambda_{\mathcal{C}}$ has the group property.

(iii) Show that $\Lambda_{\mathcal{C}}$ has the following parameters:

$$\begin{aligned}
d^2_{\min}(\Lambda_{\mathcal{C}}) &= \min\{d, 4\}; \\
K_{\min}(\Lambda_{\mathcal{C}}) &= \begin{cases} 2^d N_d, & \text{if } d < 4; \\ 2n, & \text{if } d > 4; \\ 2^d N_d + 2n, & \text{if } d = 4; \end{cases} \\
\gamma_c(\Lambda_{\mathcal{C}}) &= d^2_{\min}(\Lambda_{\mathcal{C}}) 2^{-\eta(\mathcal{C})},
\end{aligned}$$

where $\eta(\mathcal{C}) = 2(n - k)/n$ is the redundancy of $\mathcal{C}$ in bits per two dimensions.

Problem 10.2 (Invariance of nominal coding gain)

Show that $\gamma_c(\Lambda)$ is invariant to scaling, orthogonal transformations, and Cartesian products; *i.e.,* $\gamma_c(\alpha U \Lambda^m) = \gamma_c(\Lambda)$, where $\alpha > 0$ is any scale factor, U is any orthogonal matrix, and $m \geq 1$ is any positive integer. Show that $\gamma_c(\alpha U \mathbb{Z}^n) = 1$ for any version $\alpha U \mathbb{Z}^n$ of any integer lattice $\mathbb{Z}^n$.

Problem 10.3 (Invariance of normalized second moment)

Show that $G(\mathcal{R})$ is invariant to scaling, orthogonal transformations, and Cartesian products; *i.e.,* $G(\alpha U \mathcal{R}^m) = G(\mathcal{R})$, where $\alpha > 0$ is any scale factor, U is any orthogonal matrix, and $m \geq 1$ is any positive integer. Show that $G(\alpha U[-1,1)^n) = 1/12$ for any version $\alpha U[-1,1)^n$ of any n-cube $[-1,1)^n$ centered at the origin.