 Wednesday November 8, 2006

Solutions to Quiz 2

Notations: we will use $\langle\cdot,\cdot\rangle \equiv \langle\cdot,\cdot\rangle_{\mathcal{L}_2}$ and $||\cdot|| \equiv ||\cdot||_{\mathcal{L}_2}$.

Problem 1

(a) True:
Let $w = u - v$, which is continuous, and let us assume that there exists t_0 s.t. $w(t_0) \neq 0$ (say $w(t_0) > 0$ w.l.o.g). Then by continuity, there exists $\delta > 0$ such that for any $t \in (t_0 - \delta, t_0 + \delta)$, we have $w(t) \in (w(t_0) - w(t_0)/2, w(t_0) + w(t_0)/2)$. Thus $\mu\{t : w(t) \neq 0\} \geq \delta > 0$ (and $\int_{-\infty}^{+\infty} |w(t)|dt \geq \delta w(t_0)/2 > 0$), contradicting the fact that u and v are $\mathcal{L}_2$ equivalent.

(b) False:
The sampling theorem also requires that u is continuous, in order to to have a complete representation of u in terms of its samples. If one does not start with a continuous function, but simply an $\mathcal{L}_2$ function whose Fourier transform is band limited, the samples might not have anymore meaning. As soon as the functions possesses first order discontinuities (not only isolated points, but jumps), the meaning of samples is lost, as the sinc expansion is a continuous function. If the function has only isolated discontinuous points (i.e. there exist a continuous function which equals the initial one a.e.), and if one samples only at continuity points, we can still get an $\mathcal{L}_2$ equivalent representation. But as soon as a sample is taken at a discontinuity point, then again we loose track of the function. For example, consider the function $\mathcal{X}_0$, which is 1 at zero, and 0 everywhere else, it is clearly $\mathcal{L}_2$ and has a band limited Fourier transform (zero function). Then, the only non-zero sample is $u(0) = 1$, and if we tried to reconstruct the function with the sampling theorem, we would get a sinc.

(c) False:
If $\hat{u}$ is band limited to f_c, then in fact $\hat{v}(f) = 2\hat{u}(f)$ for all $|f| \leq f_c$, otherwise $\hat{v}$ can be anything.

(d) False:
We have

$$\langle x_1 + y_1, x_2 + y_2\rangle = \langle x_1, y_2\rangle + \langle y_1, x_2\rangle,$$

which does not not have to be zero (e.g. it is 2 if $x_1(t) = y_2(t) = \text{rect}(t) = x_2(t-1) = y_1(t-1)$), unless we require that $\langle x_1, y_2\rangle = -\langle y_1, x_2\rangle$.

Problem 2:

(a) Plugging $\tilde{u}(0) = 0$, we get $|u(t)-\tilde{u}(t)| = \text{sinc}(\frac{t}{T})$, for any t, and using Parseval identity, $||\text{sinc}(\frac{t}{T})||^2 = ||T\text{rect}(fT)||^2 = T$. One can equivalently use the sampling theorem energy equation, i.e. $||u - \tilde{u}||^2 = T(u(0) - \tilde{u}(0))^2 = T$.

(b) We now want to find $\arg\min_{\tilde{u}_0} \mathbb{E}|u(0) - \tilde{u}(0)|^2$, where $u(0) \sim P_{u_0}$. This problem was a central one in quantization and to see why $\bar{u}_0 \equiv \mathbb{E}_{P_{u_0}} u(0)$ is the minimizer, one can notice that

$$\mathbb{E}|u(0) - \tilde{u}(0)|^2 = \mathbb{E}|u(0) - \bar{u}_0 + \bar{u}_0 - \tilde{u}(0)|^2 = \mathbb{E}|u(0) - \bar{u}_0|^2 + |\bar{u}_0 - \tilde{u}(0)|^2.$$

(c) Using Parseval, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \text{sinc}(t)\text{sinc}(t - \frac{1}{2})dt &= \langle \text{sinc}, \tau_{1/2} \circ \text{sinc} \rangle = \langle \text{rect}(f), \text{rect}(f) e^{-2\pi i f/2} \rangle \\ &= \int_{-1/2}^{1/2} e^{-\pi i f} df = \frac{2}{\pi}. \end{aligned} \tag{1}$$

More directly as sinc is symmetric and rect $=$ rect^2, we have $\text{sinc}(t) \star \text{sinc}(-t) = \text{sinc}(t)$, and we get the answer $\text{sinc}(1/2) = \frac{2}{\pi}$.

Note: The function $t \mapsto \text{sinc}(t/T)$ is continuous and band-limited to $[-1/2T, 1/2T]$. Thus from the sampling theorem, we can express it in terms of it samples at times kT and the $\text{sinc}(t/T - k)$ basis ($k \in \mathbb{Z}$). But instead of that, we would like to express it in terms of its samples at time $kT + T/2$ using the translated functions $\text{sinc}(t/T - k - 1/2)$. To make sure that this is possible, note that $u \mapsto \text{sinc}(u/T + 1/2)$ is also continuous and band-limited to $[-1/2T, 1/2T]$, from the sampling theorem,

$$\text{sinc}(u/T + 1/2) = \sum_k \text{sinc}(k + 1/2)\text{sinc}(t/u - k), \quad \forall u,$$

and using the change of variable $t/T = u/T + 1/2$, we get

$$\text{sinc}(t/T) = \sum_k \text{sinc}(k + 1/2)\text{sinc}(t/T - k - 1/2), \quad \forall t. \tag{2}$$

Thus $\{\text{sinc}(t/T - k - 1/2)\}_k$ are in fact a spanning set of functions, and having previous change of variable in mind, they are of course orthogonal (with norm $T^{\frac{1}{2}}$). Therefore by orthogonality

$$\int_{-\infty}^{+\infty} \text{sinc}(t/T)\text{sinc}(t/T - \frac{1}{2})dt = T\text{sinc}(1/2) = \frac{2T}{\pi},$$

recovering (1) for $T = 1$.

This different way of dealing with the problem will allow us to solve more easily the next questions too, although we will keep comparing the two different ways.

(d) We want to minimize

$$||\text{sinc}(t/T) - \tilde{u}_1 \text{sinc}(t/T - 1/2)||^2.$$

A tedious way is to expand this as

$$\begin{aligned}||\text{sinc}(t/T) - \tilde{u}_1\text{sinc}(t/T - 1/2)||^2 &= ||\text{sinc}(t/T)||^2 + ||\tilde{u}_1\text{sinc}(t/T - 1/2)||^2 \\ &- 2\langle\text{sinc}(t/T), \tilde{u}_1\text{sinc}(t/T - 1/2)\rangle\end{aligned}$$

and using (c), directly tackle the minimization of $T + \tilde{u}_1^2 T - \tilde{u}_1 4T/\pi$. Since it is a convex function in $\tilde{u}_1$, we get as a minimizer $\tilde{u}_1^* = \frac{2}{\pi}$, and $T(1 - \frac{4}{\pi^2})$ for the error energy.
But what we are really doing here, is to project $\text{sinc}(t/T)$ onto the space $\{\lambda u : \lambda \in \mathbb{R}\}$, where $u(t) = \text{sinc}(t/T - 1/2)$, and we know that the projected vector will have minimal $\mathcal{L}_2$ error (by orthogonality principle). The projected vector is then $\frac{\langle\text{sinc}(t/T), u(t)\rangle}{||u(t)||^2} u(t)$, meaning that the optimal scaling is $\tilde{u}_1^* = \frac{\langle\text{sinc}(t/T), u(t)\rangle}{||u(t)||^2} = \frac{2}{\pi}$, which gives us again (by the pythagorean theorem) $T(1 - \frac{4}{\pi^2})$ for the error energy.
On the other hand, using our expansion from (2), we have

$$\begin{aligned}||\text{sinc}(t/T) - \tilde{u}_1\text{sinc}(t/T - 1/2)||^2 &= ||(\text{sinc}(1/2) - \tilde{u}_1)\text{sinc}(t/T - 1/2) \\ &+ \sum_{k\neq 0} \text{sinc}(k + 1/2)\text{sinc}(t/T - k - 1/2)||^2 \\ &= |\text{sinc}(1/2) - \tilde{u}_1|^2 ||\text{sinc}(t/T - 1/2)||^2 \\ &+ ||\sum_{k\neq 0} \text{sinc}(k + 1/2)\text{sinc}(t/T - k - 1/2)||^2\end{aligned}$$

where last equalities uses the orthogonality property. Therefore, we directly get $\tilde{u}_1^* = \text{sinc}(1/2) = \frac{2}{\pi}$

(e) We now have to minimize

$$||\text{sinc}(t/T) - \tilde{u}_1\text{sinc}(t/T - 1/2) - \tilde{u}_2\text{sinc}(t/T + 1/2)||^2,$$

we can again use the projection theorem on the two dimensional space spanned by the orthogonal vectors $\{\text{sinc}(t/T - 1/2), \text{sinc}(t/T + 1/2)\}$ getting

$$\tilde{u}_1^* = \frac{1}{T}\langle\text{sinc}(t/T), \text{sinc}(t/T - 1/2)\rangle = \tilde{u}_2^* = \frac{1}{T}\langle\text{sinc}(t/T), \text{sinc}(t/T + 1/2)\rangle = \frac{2}{\pi}.$$

Or, using (2), we get

$$\begin{aligned}&||\text{sinc}(t/T) - \tilde{u}_1\text{sinc}(t/T - 1/2) - \tilde{u}_2\text{sinc}(t/T + 1/2)||^2 \\ =\ & ||(\text{sinc}(1/2) - \tilde{u}_1)\text{sinc}(t/T - 1/2) \\ +\ & (\text{sinc}(-1/2) - \tilde{u}_2)\text{sinc}(t/T + 1/2) \\ +\ & \sum_{k \ni \{0,-1\}} \text{sinc}(k + 1/2)\text{sinc}(t/T - k - 1/2)||^2,\end{aligned} \tag{3}$$

which, by orthogonality principle, directly gives us

$$\tilde{u}_1^* = \tilde{u}_2^* = \text{sinc}(1/2) = \frac{2}{\pi}.$$

(f) Looking at (3), it is clear why using only two sinc functions will not allow us to get a zero error energy, and that in fact, infinity of them will be required (all k's in $\mathbb{Z}$) to drive it to zero.

(g) The choice of $\tilde{u}_1^*$ is the same as in (c), because the error energy is still expressed as

$$||u - \tilde{u}||^2 = ||\text{sinc}(t/T) - \tilde{u}_1 \text{sinc}(t/T - 1/2)||^2.$$

(h) We now have

$$\begin{aligned} ||u - \tilde{u}||^2 &= ||\text{sinc}(t/T) - \tilde{u}_1 \text{sinc}(t/T - 1/2) \\ &+ \sum_{k \neq 0} (u(kT) - \tilde{u}(kT)) \text{sinc}(t/T - k)||^2. \end{aligned}$$

Replacing $\text{sinc}(t/T - 1/2)$ by its sampling expansion

$$\text{sinc}(t/T - 1/2) = \sum_k \text{sinc}(k - 1/2) \text{sinc}(t/T - k), \quad \forall t,$$

the error energy becomes

$$\begin{aligned} & ||(1 - \tilde{u}_1 \text{sinc}(-1/2)) \text{sinc}(t/T) \\ + & \sum_{k \neq 0} (u(kT) - \tilde{u}(kT) - \tilde{u}_1 \text{sinc}(k - 1/2)) \text{sinc}(t/T - k)||^2 \\ = & \; T \left((1 - \tilde{u}_1 \text{sinc}(-1/2))^2 + \sum_{k \neq 0} (u(kT) - \tilde{u}(kT) - \tilde{u}_1 \text{sinc}(k - 1/2))^2 \right). \end{aligned}$$

Therefore, choosing

$$\tilde{u}_1^* = \frac{1}{\text{sinc}(1/2)} = \frac{\pi}{2}$$

and

$$\tilde{u}^*(kT) = u(kT) - \frac{\text{sinc}(k - 1/2)}{\text{sinc}(1/2)} = u(kT) - \frac{(-1)^{k+1}}{2k - 1}$$

will drive the error energy to zero.