MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Problem Set 2

Readings:
Notes from Lecture 2 and 3.

Supplementary readings:
[GS], Sections 1.4-1.7.
[C], Chapter 1.3
[W], Chapter 1.

Exercise 1. Consider a probabilistic experiment involving infinitely many coin tosses, and let $\Omega = \{0,1\}^\infty$ (think of 0 and 1 corresponding to heads and tails, respectively). A typical element $\omega \in \Omega$ is of the form $\omega = (\omega_1, \omega_2, \ldots)$, with $\omega_i \in \{0,1\}$.

As in the notes for Lecture 2, we define $\mathcal{F}_n$ as the σ-field consisting of all sets whose occurrence or nonoccurrence can be determined by looking at the result of the first n coin flips. The σ-field $\mathcal{F}$ for this model is defined as the smallest σ-field that contains all of the $\mathcal{F}_n$.

(a) Consider the event H consisting of all ω with the following property. There exists some time t at which the number of ones so far is greater than or equal to the number of zeros so far. Show that $H \in \mathcal{F}$.

(b) (Harder) Consider the set A of all ω for which the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i$$

exists. Show that $A \in \mathcal{F}$.
Note: This is important because, once we have also chosen a probability measure, it allows us to make statements about the probability that this limit (the long-term fraction of heads) exists.
Hint: The event A_x "the limit defined above exists and is equal to x" belongs to $\mathcal{F}$. However, this does not imply that $\bigcup_x A_x \in \mathcal{F}$ (why?). You need to find some other way of describing the event A in terms of unions, complements, etc., of events in the $\mathcal{F}_n$. For example, use the fact that a sequence converges if and only if it is a "Cauchy sequence."

Solution:

(a) Let $S_n = \{(\omega_1, \omega_2, \ldots) \mid \quad {}_{i=1}^{n} \omega_i \geq \lceil n/2 \rceil\}$, i.e., S_n is the set of sequences where there are at least as many ones, in the first n entries as there are zeroes. Then,

$$H = \bigcup_{n=1}^{\infty} S_n.$$

(b) Let

$$a_n = \frac{1}{n} \sum_{i=1}^{n} \omega_i.$$

According to Cauchy criterion, the sequence $\{a_n\}$ converges if and only if for any positive integer r, there exists some positive integer N such that for any $n > m > N$,

$$|a_n - a_m| < 1/r.$$

For a pair of positive integers $n > m$, we define

$$A_{1/r,n,m} = \left\{ \omega : \ \frac{1}{n} \sum_{i=1}^{n} \omega_i - \frac{1}{m} \sum_{i=1}^{m} \omega_i \ < 1/r \right\} \in \mathcal{F}_n.$$

Thus,

$$A = \bigcap_{r=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m=N}^{\infty} \bigcap_{n=m}^{\infty} A_{1/r,n,m} \in \mathcal{F}.$$

Exercise 2. Suppose that the events A_n satisfy $\mathbb{P}(A_n) \to 0$ and $\quad {}_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty$. Show that $\mathbb{P}(A_n \text{ i.o.}) = 0$. *Note:* A_n i.o., stands for "A_n occurs infinitely often", or "infinitely many of the A_n occur", or just $\limsup_n A_n$. *Hint:* Borel-Cantelli.

Solution: Define the set

$$A = \limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

We wish to show $\mathbb{P}(A) = 0$. Now, $A \subseteq \cup_{m=n}^{\infty} A_m$ for all m, and by monotonicity of the measure, $\mathbb{P}(A) \leq \mathbb{P}(\cup_{m=n}^{\infty} A_m)$, for all n. In addition,

$$\begin{aligned} \bigcup_{m=n}^{\infty} A_m &= A_n \cup (A_{n+1} \setminus A_n) \cup (A_{n+2} \setminus A_{n+1}) \cup \cdots \\ &= A_n \cup (A_{n+1} \cap A_n^c) \cup (A_{n+2} \cap A_{n+1}^c) \cup \cdots . \end{aligned}$$

Therefore, by the union bound,

$$\begin{aligned}
\mathbb{P}(A) &\leq \mathbb{P}\left(\bigcup_{m=n}^{\infty} A_m\right) \\
&\leq \mathbb{P}(A_n) + \sum_{m=n}^{\infty} \mathbb{P}(A_{m+1} \cap A_n^c).
\end{aligned}$$

This holds for all n, and therefore it holds in the limit as n goes to infinity. But the limit of the final expression is zero, since $\mathbb{P}(A_n) \to 0$, and since ${}_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty$.

Exercise 3. Consider one of our standard probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$, with $\Omega = (0, 1]$, $\mathcal{F}$ – Borel and $\mathbb{P}$ – the Lebesgue measure. To every element $\omega \in \Omega$ we assign its infinite decimal representation. We disallow decimal representations that end with an infinite string of nines. Under this condition, every number has a unique decimal representation.

(a) Let A be the set of points in $(0, 1]$ whose decimal representation contains at least one digit equal to 9. Find $\mathbb{P}[A]$.

(b) Let B be the set of points that have infinitely many 9's in the decimal representation. Find $\mathbb{P}[B]$. (Hint: Borel-Cantelli).

Solution: Part (a).

We will find the Lebesgue measure of A^c, the set of points in $(0, 1]$ whose decimal representation contains no digit equal to 9. We can scale that set (by multiplying it with a real number) to obtain the set

$$A_0 = \frac{1}{10} A^c,$$

which is the set of points in $(0, 1]$ whose decimal representation starts with a 0, and contains no digit equal to 9 afterwards. Since the set A_0 is just the same as A^c but scaled down by a factor of 10, we have that $\mathbb{P}(A_0) = \frac{1}{10}\mathbb{P}(A^c)$. Furthermore, we can do translations of that set to obtain analogous sets starting with different digits. In particular, let us define

$$A_k = \frac{k}{10} + A_0$$

as the set of points in $(0,1]$ whose decimal representation starts with a k, and has no digit equal to 9 afterwards. Note that these sets are all disjoint, and that we have

$$A^c = \bigcup_{k=0}^{8} A_k.$$

Then, using the finite additivity property of measures, and the fact that the Lebesgue measure is invariant by translations, we obtain

$$\begin{aligned}
\mathbb{P}(A^c) &= \mathbb{P}\left(\bigcup_{k=0}^{8} A_k\right) \\
&= \sum_{k=0}^{8} \mathbb{P}(A_k) \\
&= \sum_{k=0}^{8} \mathbb{P}(A_0) \\
&= \sum_{k=0}^{8} \frac{1}{10}\mathbb{P}(A^c) \\
&= \frac{9}{10}\mathbb{P}(A^c).
\end{aligned}$$

This equality can only be true if $\mathbb{P}(A^c) = 0$, and thus $\mathbb{P}(A) = 1$.

Part (b). Let B_i be the event that there is a 9 in the i-th position of the expansion. These events are independent with $\mathbb{P}(B_i) = 1/10$, for all $i \geq 1$. Thus, we have

$$\sum_{i=1}^{\infty} \mathbb{P}(B_i) = \infty.$$

Then, by Borel-Cantelli, we have

$$\mathbb{P}(B) = \mathbb{P}(\{B_i \ i.o.\}) = 1.$$

Exercise 4. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let A be an event (element of $\mathcal{F}$). Let $\mathcal{G}$ be collection of all events that are independent from A. Show that $\mathcal{G}$ need not be a σ-algebra.

Solution: $\mathcal{G}$ need not be a σ-algebra. For example, let X, Y be i.i.d., with $\mathbb{P}(X = 1) = \mathbb{P}(X = 0) = 1/2$. Let Z be the mod two sum of X and Y, so that if $X = Y$, then $Z = 0$, and if $X \neq Y$, then $Z = 1$. Then pairwise, these three random variables are independent. Let A be the event $\{Z = 1\}$. Now, the events $B_1 = \{X = 1\}$, $B_2 = \{Y = 1\}$ are both independent of A. However, $B_1 \cap B_2$ is not independent of A.

Exercise 5. Let $A_1, A_2, \ldots$ and B be events.

(a) Suppose that $A_k \searrow A$, i.e. $A_k \supset A_{k+1}$ and $A = \cap_{k=1}^{\infty} A_k$. Assume B is independent of A_k. Show that B is independent of A.

(b) Suppose that A_1 is independent of B and also that A_2 is independent of B. Is it true that $A_1 \cap A_2$ is independent of B? Prove or give a counterexample.

Solution:

(a) The sequence of events $A_k \cap B$ is decreasing and converges to the event $A \cap B$. [To see this, note that $(\cap_{k\geq 1} A_k) \cap B = \cap_{k\geq 1} (A_k \cap B)$.] Using the continuity of probability measures in the first and last equalities below, and independence in the middle equality, we have

$$\mathbb{P}(A \cap B) = \lim_{k\to\infty} \mathbb{P}(A_k \cap B) = \lim_{k\to\infty} \mathbb{P}(A_k)\mathbb{P}(B) = \mathbb{P}(A)\mathbb{P}(B).$$

(b) Consider two independent and fair coin tosses and let A_i be the event that the ith toss results in heads. Let B be the event that both tosses give the same result. It is easily checked that $\mathbb{P}(A_i \cap B) = \mathbb{P}(\{HH\}) = 1/4 = \mathbb{P}(A_i)\mathbb{P}(B)$, so that pairwise independence holds. On the other hand, $\mathbb{P}(B \mid A_1 \cap A_2) = 1 \neq \mathbb{P}(B)$. Thus, $A_1 \cap A_2$ and B are not independent.

Exercise 6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that function

$$d(A, B) \triangleq \mathbb{P}[A \triangle B]$$

satisfies the triangle inequality (i.e. $d(A, B) \leq d(A, C) + d(C, B)$ for any A, B, C).

Fun fact: Under this pseudo-metric any algebra is dense in the σ-algebra it generates. Thus, any event in a complicated σ-algebra (such as Borel) can be approximated arbitrarily well by events in a simple algebra (like finite unions of $[a, b)$).

Solution: The symmetric difference is $A\triangle B = (A\backslash B) \cup (B\backslash A)$

$$\begin{aligned}
A\triangle B &= (A\backslash B) \cup (B\backslash A) \\
&= (A \cap B^c) \cup (B \cap A^c) \\
&= (A \cap B^c \cap C) \cup (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C) \cup (B \cap A^c \cap C^c) \\
&\subset (C\backslash B) \cup (A\backslash C) \cup (C\backslash A) \cup (B\backslash C) \\
&= (A\triangle C) \cup (C\triangle B).
\end{aligned}$$

Hence, by the union bound,

$$\mathbb{P}(A \triangle B) \leq \mathbb{P}(A \triangle C) + \mathbb{P}(C \triangle B).$$

Exercise 7. ***[Optional, not to be graded]*** Let $\Omega_1 \subset \Omega$ and let $\mathcal{C}$ be some collection of subsets of Ω. Let

$$\mathcal{C}_1 = \mathcal{C} \cap \Omega_1 \triangleq \{A \cap \Omega_1 : A \in \mathcal{C}\}$$

and denote by $\mathcal{F}_1$ ($\mathcal{F}$) the minimal σ-algebra on Ω_1 (Ω) generated by $\mathcal{C}_1$ ($\mathcal{C}$). Also define

$$\mathcal{F}_2 = \mathcal{F} \cap \Omega_1 \triangleq \{A \cap \Omega_1 : A \in \mathcal{F}\}.$$

$\mathcal{F}_2$ is called *a trace* of $\mathcal{F}$ on Ω_1. Show $\mathcal{F}_1 = \mathcal{F}_2$. (*Hint:* show that collection $\mathcal{G} = \{E \in \mathcal{F} : E \cap \Omega_1 \in \mathcal{F}_1\}$ is a monotone class.)

Solution: For a collection $\mathcal{D}$ and a space Ω let $\alpha_\Omega(\mathcal{D})$ denote the smallest algebra of sets in Ω containing $\mathcal{D}$.

Claim: $\alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1) = \alpha_\Omega(\mathcal{C}) \cap \Omega_1$.

By definition $\mathcal{C} \subset \alpha_\Omega(\mathcal{C})$ and therefore, $\mathcal{C} \cap \Omega_1 \subset \alpha_\Omega(\mathcal{C}) \cap \Omega_1$. The empty set $\phi = \phi \cap \Omega_1 \in \alpha_\Omega(\mathcal{C}) \cap \Omega_1$, as $\alpha_\Omega(\mathcal{C})$ is an algebra. Let $E \cap \Omega_1 \in \alpha_\Omega(\mathcal{C}) \cap \Omega_1$, then $(E \cap \Omega_1)^c = \Omega_1 \backslash (E \cap \Omega_1) = E^c \cap \Omega_1 \in \alpha_\Omega(\mathcal{C}) \cap \Omega_1$, as $E \in \alpha_\Omega(\mathcal{C})$ and $\alpha_\Omega(\mathcal{C})$ is an algebra. Let $E_1 \cap \Omega_1, E_2 \cap \Omega_1 \in \alpha_\Omega(\mathcal{C}) \cap \Omega_1$, then $(E_1 \cap \Omega_1) \cap (E_2 \cap \Omega_2) = (E_1 \cap E_2) \cap \Omega_1 \in \alpha_\Omega(\mathcal{C}) \cap \Omega_1$, as $E_1, E_2 \in \alpha_\Omega(\mathcal{C})$ and $\alpha_\Omega(\mathcal{C})$ is an algebra. Hence $\alpha_\Omega(\mathcal{C})$ is an algebra of sets in Ω_1 containing $\mathcal{C} \cap \Omega_1$, and by minimality of $\alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1)$, $\alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1) \subset \alpha_\Omega(\mathcal{C}) \cap \Omega_1$.

Consider the set

$$\mathcal{D}_1 = \{E \in 2^\Omega \mid E \cap \Omega_1 \in \alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1)\}.$$

The collection $\mathcal{C} \subset \mathcal{D}_1$, as $\mathcal{C} \cap \Omega_1 \subset \alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1)$ by definition. The empty set $\phi \cap \Omega_1 = \phi \in \alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1)$, as α_{Ω_1} is an algebra. Thus $\phi \in \mathcal{D}_1$. Let $E \in \mathcal{D}_1$, then $E^c \cap \Omega_1 = \Omega_1 \backslash (E \cap \Omega_1) = (E \cap \Omega_1)^c \in \alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1)$, as $E \cap \Omega_1 \in \alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1)$ and $\alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1)$ is an algebra. Thus $\mathcal{D}_1$ is closed under complements. Let $E_1, E_2 \in \mathcal{D}_1$, then $(E_1 \cap E_2) \cap \Omega_1 = (E_1 \cap \Omega_1) \cap (E_2 \cap \Omega_1) \in \alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1)$, as $E_1 \cap \Omega_1, E_2 \cap \Omega_1 \in \alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1)$ and $\alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1)$ is an algebra. Thus $\mathcal{D}_1$ is closed under intersections and $\mathcal{D}_1$ is an algebra of sets in Ω containing $\mathcal{C}$. Therefore, by minimality $\alpha_\Omega(\mathcal{C}) \subset \mathcal{D}_1$. By definition of $\mathcal{D}_1$, $\alpha_\Omega(\mathcal{C}) \cap \Omega_1 \subset \alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1)$, which proves the claim.

Claim: For a collection of sets $\mathcal{D}$ and a space Ω, $\sigma_\Omega(\mathcal{D}) = \sigma_\Omega(\alpha_\Omega(\mathcal{D}))$.

By definition $\mathcal{D} \subset \alpha_\Omega(\mathcal{D}) \subset \sigma_\Omega(\mathcal{D})$, and by monotonicity of the $\sigma_\Omega(\cdot)$ operator, see recitation 2, $\sigma_\Omega(\mathcal{D}) \subset \sigma_\Omega(\alpha_\Omega(\mathcal{D})) \subset \sigma_\Omega(\sigma_\Omega(\mathcal{D})) = \sigma_\Omega(\mathcal{D})$. Thus $\sigma_\Omega(\mathcal{D}) = \sigma_\Omega(\alpha_\Omega(\mathcal{D}))$.

Combining the results of the two claims $\sigma_{\Omega_1}(\alpha_\Omega(\mathcal{C}) \cap \Omega_1) = \sigma_{\Omega_1}(\alpha_{\Omega_1}(\mathcal{C} \cap \Omega_1)) = \sigma_{\Omega_1}(\mathcal{C} \cap \Omega_1)$ and $\sigma_\Omega(\alpha_\Omega(\mathcal{C})) = \sigma_\Omega(\mathcal{C})$. Therefore, it suffices to show that $\sigma_{\Omega_1}(\alpha_\Omega(\mathcal{C}) \cap \Omega_1) = \sigma_\Omega(\alpha_\Omega(\mathcal{C})) \cap \Omega_1$. By the monotone class theorem, as $\alpha_\Omega(\mathcal{C})$ is an algebra, this holds if and only if $\mu_{\Omega_1}(\alpha_\Omega(\mathcal{C}) \cap \Omega_1) = \mu_\Omega(\alpha_\Omega(\mathcal{C})) \cap \Omega_1$. Let $\mathcal{A} := \alpha_\Omega(\mathcal{C})$.

By definition $\mathcal{A} \subset \mu_\Omega(\mathcal{A})$ and therefore, $\mathcal{A} \cap \Omega_1 \subset \mu_\Omega(\mathcal{A}) \cap \Omega_1$. Let $\{E_n \cap \Omega_1\} \in \mu_\Omega(\mathcal{A}) \cap \Omega_1$, with $(E_n \cap \Omega_1) \subset (E_{n+1} \cap \Omega_1)$. The sequence $\{E_n\}$ may not be monotone however, $E'_n = \cup_{k=1}^{n} E_n$ is monotonic and by the monotonicity of $\{E_n \cap \Omega_1\}$, $(\cup_{k=1}^{n} E_k) \cap \Omega_1 = E_n \cap \Omega_1$, i.e. $E_n \cap \Omega_1 = E'_n \cap \Omega_1$. Since $\mu_\Omega(\mathcal{A})$ is a monotone class $E'_n \nearrow E \in \mu_\Omega(\mathcal{A})$. Therefore, $E_n \cap \Omega_1 \nearrow E \cap \Omega_1 \in \mu_\Omega(\mathcal{A}) \cap \Omega_1$, this follows since $\bigcup_{n=1}^{\infty}(E_n \cap \Omega_1) = (\bigcup_{n=1}^{\infty} E_n) \cap \Omega_1 = E \cap \Omega_1$. Similarly, let $\{E_n \cap \Omega_1\} \in \mu_\Omega(\mathcal{A}) \cap \Omega_1$, with $(E_n \cap \Omega_1) \supset (E_{n+1} \cap \Omega_1)$, and, by the construction given for increasing sets, WLOG $E_n \supset E_{n+1}$. Since $\mu_\Omega(\mathcal{A})$ is a monotone class $E_n \searrow E \in \mu_\Omega(\mathcal{A})$. Therefore, $E_n \cap \Omega_1 \searrow E \cap \Omega_1 \in \mu_\Omega(\mathcal{A}) \cap \Omega_1$, this follows since $\;_{n=1}^{\infty}(E_n \cap \Omega_1) = (\;_{n=1}^{\infty} E_n) \cap \Omega_1 = E \cap \Omega_1$. Hence $\mu_\Omega(\mathcal{A}) \cap \Omega_1$ is a monotone class of sets in Ω_1 containing $\mathcal{A} \cap \Omega_1$ and by minimality $\mu_{\Omega_1}(\mathcal{A} \cap \Omega_1) \subset \mu_\Omega(\mathcal{A}) \cap \Omega_1$.

Consider the set

$$\mathcal{D}_2 = \{E \in 2^\Omega \mid E \cap \Omega_1 \in \mu_{\Omega_1}(\mathcal{A} \cap \Omega_1)\}.$$

The algebra $\mathcal{A} \subset \mathcal{D}_2$, as $\mathcal{A} \cap \Omega_1 \subset \alpha_{\Omega_1}(\mathcal{A} \cap \Omega_1)$ by definition. Let $\{E_n\}$ be an increasing sequence of sets in $\mathcal{D}_2$, then $\{E_n \cap \Omega_1\}$ is an increasing sequence of sets in $\mu_{\Omega_1}(\mathcal{A} \cap \Omega_1)$, and as $\mu_{\Omega_1}(\mathcal{A} \cap \Omega_1)$ is a monotone class, $(E_n \cap \Omega_1) \nearrow (E \cap \Omega_1) \in \mu_{\Omega_1}(\mathcal{A} \cap \Omega_1)$. Therefore, $E_n \nearrow E \in \mathcal{D}_1$. A similar argument holds for a decreasing sequence of sets. Hence $\mathcal{D}_2$ is a monotone class of sets in Ω containing $\mathcal{A}$. Therefore, by minimality $\mu_\Omega(\mathcal{A}) \subset \mathcal{D}_2$. By definition of $\mathcal{D}_2$, $\mu_\Omega(\mathcal{A}) \cap \Omega_1 \subset \mu_{\Omega_1}(\mathcal{A} \cap \Omega_1)$.

Hence $\mu_\Omega(\mathcal{A}) \cap \Omega_1 = \mu_{\Omega_1}(\mathcal{A} \cap \Omega_1)$, and thusly, $\sigma_\Omega(\mathcal{C}) \cap \Omega_1 = \sigma_{\Omega_1}(\mathcal{C} \cap \Omega_1)$ as desired.

Exercise 8. ***[Optional, not to be graded]*** Let $\Omega = [0, 1)$ and let $\mathcal{F}_0$ be the collection of finite unions $\cup_{i=1}^{N}[a_i, b_i)$ for $a_i, b_i \in [0, 1]$. For any $A \in \mathcal{F}_0$, let $\mathbb{P}[A] = 1$ if one of the $b_i = 1$, and $\mathbb{P}[A] = 0$ otherwise. In Lectures we showed that $\mathcal{F}_0$ is an algebra but not a σ-algebra.

(a) Show that $\mathbb{P}$ is a non-negative (finitely) additive set-function on $\mathcal{F}_0$.

(b) Show that $\mathbb{P}$ is not countably additive on $\mathcal{F}_0$.

Solution:

(a) For all $A \in \mathcal{F}_0$, $\mathbb{P}[A] \in \{0, 1\}$. Thus $\mathbb{P}$ is non-negative.
Let $A_1, A_2 \in \mathcal{F}_0$ be disjoint. Then $A_1 = \bigcup_{i=1}^{N_1} [a_i^{(1)}, b_i^{(1)})$ and $A_2 = \bigcup_{j=1}^{N_2} [a_j^{(2)}, b_j^{(2)})$, where WLOG the intervals are ordered and non are empty $a_1^{(m)} < b_1^{(m)} < \ldots < a_{N_m}^{(m)} < b_{N_m}^{(m)}$. As $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = \bigcup_{k=1}^{N_1+N_2} [a_k^{(3)}, b_k^{(3)})$, where $a_k^{(3)} \in \{a_i^{(1)}, a_j^{(2)}\}$ and $b_k^{(3)} \in \{b_i^{(1)}, b_j^{(2)}\}$ are the results of interleaving the two collections of intervals and are again WLOG ordered. By construction $\mathbb{P}[A_1] = 1$ if and only if $b_{N_1}^{(1)} = 1$, $\mathbb{P}[A_2] = 1$ if and only if $b_{N_2}^{(2)} = 1$ and $\mathbb{P}[A_1 \cup A_2] = 1$ if and only if $b_{N_1+N_2}^{(3)} = 1$. Moreover, $b_{N_1+N_2}^{(3)} = 1$ if and only if either $b_{N_1}^{(1)} = 1$ or $b_{N_2}^{(2)} = 1$. Suppose $b_{N_1+N_2}^{(3)} = 1$ and WLOG assume $b_{N_1}^{(1)} = 1$, then, as A_1 and A_2 are disjoint, $b_{N_2}^{(2)} \neq 1$

$$\mathbb{P}(A_1 \cup A_2) = 1 = 1 + 0 = \mathbb{P}(A_1) + \mathbb{P}(A_2).$$

Suppose $b_{N_1+N_2}^{(3)} \neq 1$ then neither $b_{N_1}^{(1)} = 1$ nor $b_{N_2}^{(2)} = 1$

$$\mathbb{P}(A_1 \cup A_2) = 0 = 0 + 0 = \mathbb{P}(A_1) + \mathbb{P}(A_2).$$

(b) Let $A_n = [0, 1 - \frac{1}{n})$. Then, for all n, $\mathbb{P}(A_n) = 0$. Moreover, $A_n \subset A_{n+1}$ and $\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n = [0, 1) \in \mathcal{F}_0$. Hence, by continuity of probability,

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} 0 = 0 \neq 1 = \mathbb{P}([0,1)) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right),$$

and $\mathbb{P}$ is not countably additive.