MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J Fall 2018

Problem Set 1

Readings:

(a) Notes from Lecture 1.

(b) Handout on background material on sets and real analysis (Recitation 1).

Supplementary readings:

[C], Sections 1.1-1.4.

[GS], Sections 1.1-1.3.

[W], Sections 1.0-1.5, 1.9.

Exercise 1.

(a) Let $\mathbb{N}$ be the set of positive integers. A function $f : \mathbb{N} \to \{0, 1\}$ is said to be *periodic* if there exists some N such that $f(n + N) = f(n)$, for all $n \in \mathbb{N}$. Show that the set of periodic functions is countable.

(b) Does the result from part (a) remain valid if we consider rational-valued periodic functions $f : \mathbb{N} \to \mathbb{Q}$?

Solution:

(a) For a given positive integer N, let A_N denote the set of periodic functions with a period of N. For a given N, since the sequence, $f(1), \cdots, f(N)$, actually defines a periodic function in A_N, we have that each A_N contains 2^N elements. For example, for $N = 2$, there are four functions in the set A_2:

$$f(1)f(2)f(3)f(4)\cdots = 0000\cdots; \quad 1111\cdots; \quad 0101\cdots; \quad 1010\cdots.$$

The set of periodic functions from $\mathbb{N}$ to $\{0, 1\}$, A, can be written as,

$$A = \bigcup_{N=1}^{\infty} A_N.$$

Since the union of countably many finite sets is countable, we conclude that the set of periodic functions from $\mathbb{N}$ to $\{0, 1\}$ is countable.

(b) Still, for a given positive integer N, let A_N denote the set of periodic functions with a period N. For a given N, since the sequence, $f(1), \cdots, f(N)$, actually defines a periodic function in A_N, we conclude that A_N has the same cardinality as $\mathbb{Q}^N$ (the Cartesian product of N sets of rational numbers). Since $\mathbb{Q}$ is countable, and the Cartesian product of finitely many countable sets is countable, we know that A_N is countable, for any given N. Since the set of periodic functions from $\mathbb{N}$ to $\mathbb{Q}$ is the union of $A_1, A_2, \cdots$, it is countable, because the union of countably many countable sets is countable.

Exercise 2. Let $\{x_n\}$ and $\{y_n\}$ be real sequences that converge to x and y, respectively. Provide a formal proof of the fact that $x_n + y_n$ converges to $x + y$.

Solution: Fix some $\epsilon > 0$. Let n_1 be such that $|x_n - x| < \epsilon/2$, for all $n > n_1$. Let n_2 be such that $|y_n - y| < \epsilon/2$, for all $n > n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then, for all $n > n_0$, we have

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves the desired result.

Exercise 3. We are given a function $f : A \times B \to \mathbb{R}$, where A and B are nonempty sets.

(a) Assuming that the sets A and B are finite, show that

$$\max_{x \in A} \min_{y \in B} f(x, y) \leq \min_{y \in B} \max_{x \in A} f(x, y).$$

(b) For general nonempty sets (not necessarily finite), show that

$$\sup_{x \in A} \inf_{y \in B} f(x, y) \leq \inf_{y \in B} \sup_{x \in A} f(x, y).$$

Solution:

(a) The proof rests on the application of the following simple fact: if $h(z) \leq g(z)$ for all z in some finite set Z, then

$$\min_{z \in Z} h(z) \leq \min_{z \in Z} g(z) \tag{1}$$

$$\max_{z \in Z} h(z) \leq \max_{z \in Z} g(z). \tag{2}$$

Observe that for all x, y,

$$f(x, y) \leq \max_{x \in A} f(x, y),$$

and Eq. (1) implies that for each x,

$$\min_{y\in B} f(x,y) \le \min_{y\in B} \max_{x\in A} f(x,y).$$

Now applying Eq. (2), let's take a maximum of both sides with respect to $x \in A$. Since the right-hand side is a number, it remains unchanged:

$$\max_{x\in A} \min_{y\in B} f(x,y) \le \min_{y\in B} \max_{x\in A} f(x,y),$$

which is what we needed to show.

(b) Along the same lines, we have the fact that if $h(z) \le g(z)$ for all $z \in Z$,

$$\inf_{z\in Z} h(z) \le \inf_{z\in Z} g(z) \tag{3}$$

$$\sup_{z\in Z} h(z) \le \sup_{z\in Z} g(z). \tag{4}$$

These follow immediately from the definitions of $\sup$ and $\inf$.

As before, we begin with

$$f(x,y) \le \sup_{x\in A} f(x,y),$$

for all x, y. By Eq. (3), for each x,

$$\inf_{y\in B} f(x,y) \le \inf_{y\in B} \sup_{x\in A} f(x,y),$$

and using Eq. (4),

$$\sup_{x\in A} \inf_{y\in B} f(x,y) \le \inf_{y\in B} \sup_{x\in A} f(x,y).$$

Exercise 4. A probabilistic experiment involves an infinite sequence of trials. For $k = 1, 2, \ldots$, let A_k be the event that the kth trial was a success. Write down a set-theoretic expression that describes the following event:

B: For every k there exists an ℓ such that trials $k\ell$ and $k\ell^2$ were both successes.

Note: A "set theoretic expression" is an expression like $\bigcup_{k>5} \bigcap_{\ell<k} A_{k+\ell}$.

Solution: $B = \bigcap_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} (A_{k\ell} \cap A_{k\ell^2})$.

Exercise 5. Let $f_n, f, g : [0,1] \to [0,1]$ and $a, b, c, d \in [0,1]$. Derive the following set theoretic expressions:

(a) Show that

$$\{x \in [0,1] \mid \sup_n f_n(x) \leq a\} = \bigcap_n \{x \in [0,1] \mid f_n(x) \leq a\},$$

and use this to express $\{x \in [0,1] \mid \sup_n f_n(x) < a\}$ as a countable combination (countable unions, countable intersections and complements) of sets of the form $\{x \in [0,1] \mid f_n(x) \leq b\}$.

(b) Express $\{x \in [0,1] \mid f(x) > g(x)\}$ as a countable combination of sets of the form $\{x \in [0,1] \mid f(x) > c\}$ and $\{x \in [0,1] \mid g(x) < d\}$.

(c) Express $\{x \in [0,1] \mid \limsup_n f_n(x) \leq c\}$ as a countable combination of sets of the form $\{x \in [0,1] \mid f_n(x) \leq c\}$.

(d) Express $\{x \in [0,1] \mid \lim_n f_n(x) \text{ exists}\}$ as a countable combination of sets of the form $\{x \in [0,1] \mid f_n(x) < c\}$, $\{x \in [0,1] \mid f_n(x) > c\}$, etc. (Hint: think of $\{x \in [0,1] \mid \limsup_n f_n(x) > \liminf_n f_n(x)\}$).

Solution: First observe the following set relations

$$[0,c) = \bigcup_{n=1}^{\infty} [0, c - \tfrac{1}{n}] \quad [0,c] = \bigcap_{n=1}^{\infty} [0, c + \tfrac{1}{n})$$
$$(c,1] = \bigcup_{n=1}^{\infty} [c + \tfrac{1}{n}, 1] \quad [c,1] = \bigcap_{n=1}^{\infty} (c - \tfrac{1}{n}, 1].$$

All conversions between strict and non-strict inequalities following from these relations and properties of the inverse image, i.e. homomorphism of arbitrary set operations. We will use the shorthand notation

$$\{f < a\} := \{x \in [0,1] \mid f(x) < a\}.$$

(a) Let $x \in \bigcap_n \{f_n \leq a\}$. Then, $f_n(x) \leq a$ for all $n \implies \sup_n f_n(x) \leq a$, by definition of sup as a is an upper bound for $\{f_n(x)\}$. Therefore, as x was arbitrary,

$$\bigcap_{n=1}^{\infty} \{f_n \leq a\} \subset \{\sup_n f_n \leq a\}.$$

Let $x \in \{\sup_n f_n \leq a\}$. Then $\sup_n f_n(x) \leq a$ and for all n $f_n(x) \leq \sup_n(x) \leq a$. Therefore, as x was arbitrary,

$$\{\sup_n f_n \leq a\} \subset \mathop{}_{n=1}^{\infty} \{f_n \leq a\}.$$

Hence $\{\sup_n f_n \leq a\} = \bigcap_n \{f_n \leq a\}$. By De Morgan's this relation also implies

$$\{\sup_n f_n > a\} = \bigcup_{n=1}^{\infty} \{f_n \geq a\}.$$

Similar results hold for $\inf$.

Let $f = \sup_n f_n$. Using the above comment

$$\begin{aligned}
\{\sup_n f_n < a\} &= \{f < a\} \\
&= f^{-1}([0, a)) \\
&= \bigcup_{k=1}^{\infty} f^{-1} \quad [0, a - \tfrac{1}{k}] \\
&= \bigcup_{k=1}^{\infty} \{\sup_n f_n \leq a - \tfrac{1}{k}\} \\
&= \bigcup_{k=1}^{\infty} \mathop{}_{n=1}^{\infty} \{f_n \leq a - \tfrac{1}{k}\}.
\end{aligned}$$

(b) Using countability and density of the rationals

$$\begin{aligned}
\{f > g\} &= \bigcup_{q \in \mathbb{Q}} \{f > q\} \cap \{q > g\} \\
&= \bigcup_{q \in \mathbb{Q}} \{f > q\} \cap \{q \geq g\} \\
&= \bigcup_{q \in \mathbb{Q}} \{f \geq q\} \cap \{q > g\}.
\end{aligned}$$

(c)

$$
\begin{aligned}
\{\limsup_{n\to\infty} f_n \le c\} &= \{\inf_{n\ge 1} \sup_{k\ge n} f_k \le c\} \\
&= \mathop{}\limits_{m=1}^{\infty} \{\inf_{n\ge 1} \sup_{k\ge n} f_k < c + \tfrac{1}{m}\} \\
&= \mathop{}\limits_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \{\sup_{k\ge n} f_k < c + \tfrac{1}{m}\} \\
&= \mathop{}\limits_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{\ell=1}^{\infty} \{\sup_{k\ge n} f_k \le c + \tfrac{1}{m} - \tfrac{1}{\ell}\} \\
&= \mathop{}\limits_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{\ell=1}^{\infty} \mathop{}\limits_{k=n}^{\infty} \{f_k \le c + \tfrac{1}{m} - \tfrac{1}{\ell}\}.
\end{aligned}
$$

(d)

$$
\begin{aligned}
\{\lim_{n\to\infty} f_n \ \text{ exists}\} &= \{\liminf_{n\to\infty} f_n = \limsup_{n\to\infty} f_n\} \\
&= \{\liminf_{n\to\infty} f_n < \limsup_{n\to\infty} f_n\}^c \quad (\liminf_{n\to\infty} f_n(x) \le \limsup_{n\to\infty} f_n(x)) \\
&= \left(\bigcup_{q\in\mathbb{Q}} \{\liminf_{n\to\infty} f_n < q\} \cap \{q < \limsup_{n\to\infty} f_n\} \right)^c \quad \text{(part } b\text{)} \\
&= \mathop{}\limits_{q\in\mathbb{Q}} \{\liminf_{n\to\infty} f_n \ge q\} \cup \{\limsup_{n\to\infty} f_n \le q\}.
\end{aligned}
$$

The sets $\{\liminf_{n\to\infty} f_n \ge q\}$ and $\{\limsup_{n\to\infty} f_n \le q\}$ can be expressed as countable combinations using part (c) and the fact that

$$
\begin{aligned}
-\limsup_{n\to\infty} f_n(x) &= -\inf_{n\ge 1} \sup_{k\ge n} f_k(x) \\
&= \sup_{n\ge 1} \inf_{k\ge n} (-f_k(x)) \\
&= \liminf_{n\to\infty} (-f_n(x)),
\end{aligned}
$$

i.e. $\{\liminf_{n\to\infty} f_n \ge q\} = \{\limsup_{n\to\infty}(-f_n) \le -q\}$. More specifically,

$$
\bigcap_{q\in\mathbb{Q}} \left[\left(\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{\ell=1}^{\infty} \bigcap_{k=n}^{\infty} \{f_k \ge c - \tfrac{1}{m} + \tfrac{1}{\ell}\} \right) \cup \left(\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{\ell=1}^{\infty} \bigcap_{k=n}^{\infty} \{f_k \le c + \tfrac{1}{m} - \tfrac{1}{\ell}\} \right) \right].
$$

Using one of the later two expressions of part (b), we can drop one of the outer intersections

$$\bigcap_{q\in\mathbb{Q}}\left[\left(\bigcup_{n=1}^{\infty}\bigcup_{\ell=1}^{\infty}\bigcap_{k=n}^{\infty}\{f_k \geq c+\tfrac{1}{\ell}\}\right)\cup\left(\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcup_{\ell=1}^{\infty}\bigcap_{k=n}^{\infty}\{f_k \leq c+\tfrac{1}{m}-\tfrac{1}{\ell}\}\right)\right]$$

or

$$\bigcap_{q\in\mathbb{Q}}\left[\left(\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcup_{\ell=1}^{\infty}\bigcap_{k=n}^{\infty}\{f_k \geq c-\tfrac{1}{m}+\tfrac{1}{\ell}\}\right)\cup\left(\bigcup_{n=1}^{\infty}\bigcup_{\ell=1}^{\infty}\bigcap_{k=n}^{\infty}\{f_k \leq c-\tfrac{1}{\ell}\}\right)\right].$$

Exercise 6. Let $\Omega = \mathbb{N}$ (the positive integers), and let $\mathcal{F}_0$ be the collection of subsets of Ω that either have finite cardinality or their complement has finite cardinality. For any $A \in \mathcal{F}_0$, let $\mathbb{P}(A) = 0$ if A is finite, and $\mathbb{P}(A) = 1$ if A^C is finite.

(a) Show that $\mathcal{F}_0$ is a field but not a σ-field.

(b) Show that $\mathbb{P}$ is finitely additive on $\mathcal{F}_0$; that is, if $A, B \in \mathcal{F}_0$, and A, B are disjoint, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

(c) Show that $\mathbb{P}$ is not countably additive on $\mathcal{F}_0$; that is, construct a sequence of disjoint sets $A_i \in \mathcal{F}_0$ such that $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_0$ and $\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) \neq \ {}_{i=1}^{\infty} \mathbb{P}\left(A_i\right)$.

(d) Construct a decreasing sequence of sets $A_i \in \mathcal{F}_0$ such that $\cap_{i=1}^{\infty} A_i = \emptyset$ for which $\lim_{i\to\infty} \mathbb{P}(A_i) \neq 0$.

Solution:

(a) The empty set has zero cardinality, and therefore belongs to $\mathcal{F}_0$. Furthermore, if $A \in \mathcal{F}_0$, then either A or A^c has finite cardinality. It follows that either A^c or $(A^c)^c$ has finite cardinality, so that $A^c \in \mathcal{F}_0$.

Suppose that $A, B \in \mathcal{F}_0$. If both A and B are finite, then $A \cup B$ is also finite and belongs to $\mathcal{F}_0$. Suppose now that at least one of A or B is infinite. We have $A \cup B = (A^c \cap B^c)^c$. Since $A^c \cap B^c$ is finite, it follows that $A \cup B \in \mathcal{F}_0$. This shows that $\mathcal{F}_0$ is a field.

To see that $\mathcal{F}_0$ is not a σ-field, note that $\{2n\} \in \mathcal{F}_0$ for every $n \in \mathbb{N}$, but the set $\bigcup_{n=0}^{\infty}\{2n\}$, the set of even natural numbers, is not in $\mathcal{F}_0$.

(b) Let $A, B \in \mathcal{F}_0$ be disjoint. If both A and B are finite, then $\mathbb{P}\left(A \cup B\right) = 0 = \mathbb{P}(A) + \mathbb{P}(B)$. Suppose that either A or B (or both) is infinite. Since A and B are disjoint, we have $A \subset B^c$ and $B \subset A^c$. It follows that A and B cannot both be infinite. Therefore, $\mathbb{P}(A \cup B) = 1 = \mathbb{P}(A) + \mathbb{P}(B)$, and $\mathbb{P}$ is finitely additive.

(c) Note that $\{n\} \in \mathcal{F}_0$ and $\bigcup_{n\geq 1}\{n\} = \Omega$. However, $\mathbb{P}(\{n\}) = 0$ while $\mathbb{P}(\Omega) = 1$, hence $\mathbb{P}$ is not countably additive.

(d) Let $A_n = \{n, n+1, \ldots\}$. Then $(A_n)_{n\geq 1}$ forms a decreasing sequence of sets with $\bigcap_n A_n = \emptyset$. But $\mathbb{P}(A_n) = 1$ for all n, hence $\lim_{n\to\infty} \mathbb{P}(A_n) = 1$.