MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Problem Set 12 (OPTIONAL)

Readings:
Notes from Lectures 23-26.
[Cinlar], Chapter V.1-V.4
[Grimmett-Stirzaker], Chapter 6.1-6.6, Chapter 12.1-12.6

Exercise 1. Consider two irreducible Markov chains K_1 and K_2 on $\mathbb{Z}_+$ whose only jumps are of the form $n \to \{n-1, n, n+1\}$. Suppose

$$K_1(i, \{i+1\}) \geq K_2(j, \{j+1\}) \qquad \forall i, j \geq 0$$

and

$$K_1(i, \{i-1\}) \leq K_2(i, \{i-1\}) \qquad \forall i \geq 0.$$

Show that there exists a coupling such that $X_n \geq Y_n$ a.s.. Conclude that if Y is transient, then so is X.

Solution: Start the two Markov chains in some arbitrary state i_0. Evolve X_n according to K_1 and Y_n as follows. Suppose $X_n = i$. Let

$$p_{-1} = K_1(i, \{i-1\}) \qquad p_0 = K_1(i, \{i\}) \qquad p_1 = K_1(i, \{i+1\}),$$

$$q_{-1} = K_2(i, \{i-1\}) \qquad q_0 = K_2(i, \{i\}) \qquad q_1 = K_2(i, \{i+1\}).$$

By assumption $p_1 \geq q_1$ and $p_{-1} \leq q_{-1}$. There are two cases $p_0 \geq q_0$ and $p_0 \leq q_0$. Define $\mathbb{P}(\underline{X}_{n+1}) := (\mathbb{P}(X_{n+1} = i+1), \mathbb{P}(X_{n+1} = i), \mathbb{P}(X_{n+1} = i-1))$ and $\mathbb{P}(\underline{Y}_{n+1}) := (\mathbb{P}(Y_{n+1} = i+1), \mathbb{P}(Y_{n+1} = i), \mathbb{P}(Y_{n+1} = i-1))$. Let $\mathbb{P}(\underline{Y}_{n+1}) = A\mathbb{P}(\underline{X}_{n+1})$ where the corresponding matrices for the two case are respectively

$$\begin{bmatrix} \frac{q_1}{p_1} & 0 & 0 \\ 0 & \frac{q_0}{p_0} & 0 \\ 1 - \frac{q_1}{p_1} & 1 - \frac{q_0}{p_0} & 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{q_1}{p_1} & 0 & 0 \\ \frac{q_0 - p_0}{p_1} & 1 & 0 \\ \frac{q_{-1} - p_{-1}}{p_1} & 0 & 1 \end{bmatrix}.$$

Exercise 2. Consider a Bernoulli process $(X_n, n \geq 1)$. For each of the integer valued random variables T below determine whether it is a stopping time or not. In case T is a stopping time describe the corresponding sequence of functions $h_n = h_n(x_1, \ldots, x_n)$ determining T.

(a) T is the first time n such that $\sum_{1\le i\le n} X_i = 2$. Namely, $T = \min\{n : {}_{1\le i\le n} X_i = 2\}$. If T is indeed a stopping time describe the corresponding sequence of functions h_n.

(b) $T = \max(10, \min\{n : \sum_{1\le i\le n} X_i = 2\})$.

(c) T is the first time n such that $\sum_{1\le i\le n} X_i = \sum_{n+1\le i\le 2n} X_i$.

(d) T is the first time n such that $\sum_{1\le i\le n/2} X_i = \sum_{n/2+1\le i\le n-1} X_i$.

Solution:

(a) Yes, it is a stopping time.

$$h_n(x_1,\dots,x_n) = \mathbb{1}(x_1+\cdots+x_{n-1} < 2, x_1+\cdots x_n = 2).$$

(b) Yes, it is a stopping time. $h_n(x_1,\dots,x_n) =$

$$\mathbb{1}(n = 10, x_1+\cdots+x_{10} \ge 2) + \mathbb{1}(x_1+\cdots+x_{n-1} < 2, x_1+\cdots x_n = 2, n > 10).$$

(c) No, it is not a stopping time because it depends on the future (entries in $n+1,\dots,2n$).

(d) Yes it is a stopping time.

$$h_n(x_1,\dots,x_n) = \mathbb{1}\left(\sum_{i=1}^{\lfloor k/2\rfloor} x_i \ne \sum_{\lfloor k/2\rfloor+1}^{k-1} x_i \quad k < n-1, \sum_{i=1}^{\lfloor n/2\rfloor} x_i = \sum_{\lfloor n/2\rfloor+1}^{n-1} x_i\right)$$

Exercise 3. Let X_1, X_2, X_3 be independent exponential random variables with mean 1. Let

$$\alpha = \mathbb{P}(X_1 > X_2 + X_3).$$

(a) Find α, without calculating any integrals.

(b) Find the probability that the largest of the three random variables X_1, X_2, X_3 is larger than the sum of the other two. [You can express your answer in terms of the constant α from part (a).]

Solution:

(a) Consider two independent rate one Poisson processes, say A and B. Let X_2 be the time until the first arrival of A, let X_3 be the time between the first and second arrival of A, and let X_1 be the time of the first arrival of B. Then α is the probability that A has two arrivals before B. By Poisson splitting/merging, we can instead consider a single rate two Poisson process, where we assign arrivals to A or B i.i.d. with probability $1/2$. Thus α is the probability that the first two arrivals of the merged process both go to A, thus $\alpha = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

(b) As the events $\{X_1 > X_2 + X_3\}$, $\{X_2 > X_1 + X_3\}$, and $\{X_3 > X_1 + X_2\}$ are disjoint, and there union is the event that the largest X_i is bigger than the sum of the smaller three, we have

$$\begin{aligned}
&\mathbb{P}(\{X_1 > X_2 + X_3\} \cup \{X_2 > X_1 + X_3\} \cup \{X_3 > X_1 + X_2\}) \\
&\qquad = \mathbb{P}(X_1 > X_2 + X_3) + \mathbb{P}(X_2 > X_1 + X_3) + \mathbb{P}(X_3 > X_1 + X_2) \\
&\qquad = 3\mathbb{P}(X_1 > X_2 + X_3) \\
&\qquad = 3\alpha.
\end{aligned}$$

where second equality follows by symmetry as the variables are i.i.d.

Exercise 4. Fast and slow customers arrive at a 24 hour store according to independent Poisson processes, each with rate 1 per minute. Fast customers stay in the bookstore for 1 minute, slow customers stay in the store for 2 minutes.

(a) What is the PMF of the total number of customer arrivals during a one minute interval?

(b) Find the variance of the number of customers in the store at 3 p.m.

(c) At 3 p.m., there is only one customer present in the store.

(i) What is the probability, β, that the customer is a fast one?

(ii) What is the PDF that this customer will depart before a new customer arrives? [You may express your answer in terms of the constant β from part (i). Also, you may leave your answer as a formula involving integrals – you do not have to evaluate the integrals.]

Let N_t be the number of fast customer arrivals during $[0, t]$.

(d) Does $(N_{2t} - N_t)/t$ converge in probability, as $t \to \infty$? With probability 1? If yes, to what? Outline a rigorous justification for your answers. You can start with t integer-valued and then argue for $t \in \Re$.

(e) Find (approximately) a time k such that

$$\mathbb{P}(N_k \geq 100) \approx 0.758.$$

Note that if Z is a standard normal random variable, then $\mathbb{P}(Z \leq 0.7) = 0.758$. [You do not need to be rigorous in deriving your answer. You may leave your answer in the form of an equation for k, which you do not need to solve numerically.]

Solution:

(a) By Poisson merging, the total number of arrivals is a Poisson process with a rate of two customers per minute, so in an one minute window there are $\text{Pois}(2)$ arrivals.

(b) A fast customer will be in the store at 3pm iff he arrived between 2:59pm and 3:00pm, because they only stay for a minute. Thus there will be $\text{Pois}(1)$ fast customers in the store. Likewise a slow customer will be in the store iff he arrived between 2:58pm and 3:00pm, thus there will be $\text{Pois}(2)$ slow customers in the store. As the sum of $\text{Pois}(\lambda)$ and $\text{Pois}(\mu)$ independent is $\text{Pois}(\lambda + \mu)$, the total number of customers in the store at 3:00pm will be $\text{Pois}(3)$. This random variable has variance 3.

(c) (i) By Poisson merging, we can instead suppose that we drew $\text{Pois}(3)$ total customers to be in the store, and then for each customer, with probability $1/3$ assigned them as "fast" and with probability $2/3$ assigned them as slow, i.d.d. and independent of the total number of customers. Thus by independence of the number of customers and their classification, conditional on the total number of customers being one, the probability the customer will be fast is $\beta = 1/3$.

(ii) Let Z be the amount of time that the customer in the store will remain before departing. Recall that for a Poisson process conditioned to have k arrivals in $[0, t]$, the time of each arrival is i.i.d. Uni$(0, t)$. Thus the distribution of the arrival time is Uni$(0, 1)$ when the customer was fast and Uni$(0, 2)$ when the customer was slow. Thus the distribution of Z is Uni$(0, 1)$ if the customer was fast and Uni$(0, 2)$ if the customer was slow. Given that $Z = z$, the probability that we have a new arrival before the customer departs is $\mathbb{P}(\text{Exp}(2) < z)$. Thus by total probability,

$$\mathbb{P}(\text{arrival before departure}) = \beta \int_0^1 \exp(-2z)dz + (1-\beta) \int_0^2 \frac{1}{2} \exp(-2z)dz.$$

(d) Assume t is integer. Let $X_k = N_k - N_{k-1}$ for k integer, so X_k are i.i.d $\mathrm{Pois}(1)$. We have that $N_{2t} - N_t =_d N_t = \;_{k=1}^{t} X_k$ by Lecture 21 page 5 property b. Thus for all $\epsilon > 0$,

$$\lim_{t\to\infty} \mathbb{P}\left(\left|\frac{N_{2t}}{t} - \frac{N_t}{t} - 1\right| > \epsilon\right) = \lim_{t\to\infty} \mathbb{P}\left(\left|\frac{1}{t}\sum_{i=1}^{t} X_i - 1\right| > \epsilon\right) = 0,$$

by the WLLN. This shows the result for integer t. The assumption that t was integer was not essential. Suppose we take $t \to \infty$ along $t_n = cn$ for any $c > 0$, where n is integer. Then we can divide N_t into intervals of length c making $\tilde{X}_k \sim \mathrm{Pois}(c)$ be i.i.d. so then

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{N_{2t_n}}{t_n} - \frac{N_{t_n}}{t_n} - 1\right| > \epsilon\right) &= \lim_{n\to\infty} \mathbb{P}\left(\left|\frac{1}{cn}\sum_{i=1}^{n} \tilde{X}_i - 1\right| > \epsilon\right) \\ &= \lim_{n\to\infty} \mathbb{P}\left(\left|\frac{1}{n}\sum^{n} \tilde{X}_i - c\right| > \epsilon c\right) = 0\end{aligned}$$

by the WLLN.

(e) As in the previous problem for integer k, $N_k = \sum_{i=1}^{k} X_i$, where $X_i \sim \mathrm{Pois}(1)$ are i.i.d., and thus $\mathbb{E}[X_1] = \mathrm{var}(X_1) = 1$. Thus by the CLT,

$$\begin{aligned}\mathbb{P}(N_k \geq 100) = \mathbb{P}\left(\sum_{i=1}^{k} X_i \geq 100\right) &= \mathbb{P}\left(\frac{1}{\sqrt{k}}\sum_{i=1}^{k}(X_i - 1) \geq \frac{100-k}{\sqrt{k}}\right) \\ &\approx \mathbb{P}\left(\mathrm{N}(0,1) \geq \frac{100-k}{\sqrt{k}}\right).\end{aligned}$$

Thus we need k that solves

$$\frac{100-k}{\sqrt{k}} = -0.7.$$

Exercise 5. Let S be the set of arrival times in a Poisson process on $\mathbb{R}$ (i.e., a process that has been running forever), with rate λ. Each arrival time in S is displaced by a random amount. The random displacement associated with each element of S is a random variable that takes values in a finite set. We assume that the random displacements associated with different arrivals are independent and identically distributed. Show that the resulting process (i.e., the process whose arrival times are the displaced points) is a Poisson process with rate λ. (We expect a proof consisting of a verbal argument, using known properties of Poisson processes; formulas are not needed.)

Solution: Let $\{v_1, \ldots, v_m\}$ be the set of values that our perturbations can take, and let $\{p_1, \ldots, p_m\}$ be the probabilities of each perturbation outcome. As the perturbation are independent of the Poisson process, by Poisson splitting this is equivalent to m independent Poisson processes, with rate λp_i for $i = 1, \ldots, m$, where the ith process has every point translated by v_i. But by the stationarity property of Poisson process (property b, page 5 of lecture 21), the distribution of the number of points on any interval for each of the i independent and translated processes will be the same as if we did not apply the translation, since the length of the interval determines the distribution of the number of points. Finally, we can apply Poisson merging to the untranslated processes (which are still independent), to recover a Poisson process with rate λ.

Exercise 6.

(a) Consider a Markov chain with several recurrent classes $R_l, 1 \leq l \leq L$. For each $l = 1, 2, \ldots, L$ consider the distribution $\pi_l = (\pi_x^l, 1 \leq x \leq N)$ on the entire states space defined as follows. For each $x \in R_l$, $\pi_x = 1/\mu_x$, where μ_x is the mean recurrence time of state x, and $\pi_y = 0$ for all $y \notin R_l$. Show that π_l is a stationary distribution.

(b) Suppose π and μ are stationary distribution of some Markov chain X_n. Prove that any convex combination $\nu = \lambda\pi + (1-\lambda)\mu$ is also a stationary distribution. Namely $\lambda \in [0, 1]$ and ν is defined by $\nu_i = \lambda\pi_i + (1-\lambda)\mu_i$.

From parts (a) and (b) we conclude that a Markov chain with several recurrence classes has infinitely many stationary distribution. One can show that every stationary distribution can be obtained in the way described by (b), namely as a convex combination $\sum_{1 \leq l \leq L} \lambda_l \pi_l$ of stationary distributions described in part (a).

Solution:

(a) Let P be the transition matrix of X_n. Fix some recurrent class. WLOG, assume that the states of the Markov chain are $\{1, \ldots, n\}$ and the recurrent class contains states $\{1, \ldots, k\}$, with $k < n$. Let Q the top left k by k block of P. In particular, let P have the block decomposition

$$P = \begin{bmatrix} Q & 0 \\ R & S \end{bmatrix}$$

where the top right corner is all zeros because the states $\{1, \ldots, k\}$ never transition outside of $\{1, \ldots, k\}$.

Create a new Markov chain that only contains the states $\{1,\ldots,k\}$, and the edges internal to the recurrent class, with the same transition probabilities, (so with transition matrix Q). First, we observe that as the set of states is a recurrent class, every state in the class can only point to other states in the class, thus the transition probabilities leaving each state sum to one and our new structure is indeed a Markov chain. Next, as the new Markov chain has a single recurrent class, it has a unique stationary distribution $\pi = (\pi_1,\ldots,\pi_k)$ satisfying $\pi' Q = \pi'$. By theorem 1 from lecture 23, $\pi_i = 1/\mu_i$, where μ_i is the mean recurrence time. (The assumption of aperiodicity is only needed to show that the transient distribution converges to the steady state distribution.) However, the mean recurrence time of state i in the new Markov chain will be same as the mean recurrence time of state i in the original Markov chain, as by a coupling argument we can make there recurrence times equal surely. Let $\bar{\pi}$ be an n dimensional vector such that

$$\bar{\pi}_i = \begin{cases} \pi_i & i \leq k \\ 0 & i > k. \end{cases}$$

Then $\bar{\pi}$ is of the form in the claim, so it suffices to prove that $\bar{\pi}' P = \bar{\pi}'$. Let Q_i be the ith column of Q, R_i be the ith column of R, S_i be the ith column of S. Then the ith column of P for $i \leq k$ is (Q_i, R_i), so

$$\bar{\pi}' P_i = \pi' Q_i + 0 R_i = \pi_i = \bar{\pi}_i$$

and the ith column of P for $i > k$ is given by $(0, S_i)$, so

$$\bar{\pi}' P_i = \pi' 0 + 0 S_i = 0 = \bar{\pi}_i.$$

Thus $\bar{\pi} P = \bar{\pi}$, so $\bar{\pi}$ is stationary. This shows the claim.

(b) Let P be the transition matrix for X_n. As π and μ are stationary, we have that $\pi' P = \pi'$ and $\mu' P = \mu'$. Thus for $\lambda \in [0,1]$, if $\nu = \lambda\pi + (1-\lambda)\mu$, then by the linearity,

$$\nu' P = (\lambda\pi + (1-\lambda)\mu)' P = \lambda\pi' P + (1-\lambda)\mu' P = \lambda\pi' + (1-\lambda)\mu' = \nu',$$

thus ν is stationary.

Exercise 7. Consider a Markov chain $\{X_k\}$ on the state space $\{1,\ldots,n\}$, and suppose that whenever the state is i, a reward $g(i)$ is obtained. Let R_k be the total reward obtained over the time interval $\{0, 1, \ldots, k\}$, that is, $R_k = g(X_0) + g(X_1) + \cdots + g(X_k)$. For every state i, let

$$m_k(i) = \mathbb{E}[R_k \mid X_0 = i],$$

and

$$v_k(i) = \text{var}(R_k \mid X_0 = i)$$

be the mean and variance, respectively of R_k, conditioned on the initial state being equal to i.

(a) Find a recursion that given the values of $m_k(1), \ldots, m_k(n)$ allows the computation of $m_{k+1}(1), \ldots, m_{k+1}(n)$.

(b) Find a recursion that given the values of $v_k(1), \ldots, v_k(n)$ allows the computation of $v_{k+1}(1), \ldots, v_{k+1}(n)$. *Hint:* The following formula (the "law of total variance") may be useful:

$$\text{var}(X) = \mathbb{E}[\text{var}(X \mid Y)] + \text{var}(\mathbb{E}[X \mid Y]).$$

Here, $\text{var}(X \mid Y)$ stands for the variance of the conditional distribution of X given Y, and is itself a random variable because it is a function of Y.

Solution:

(a) We have $m_{k+1}(i) = \mathbb{E}[R_{k+1}|X_0 = i]$. Using the total expectation theorem, we have:

$$\begin{aligned} m_{k+1}(i) &= \mathbb{E}[g(X_0) + \cdots + g(X_{k+1})|X_0 = i] \\ &= g(i) + \sum_k p_{ij}\mathbb{E}[g(X_1) + \cdots + g(X_{k+1})|X_1 = j] \\ &= g(i) + \sum_j p_{ij} m_k(j). \end{aligned}$$

(b) Let $Q = g(X_1) + \cdots + g(X_{k+1})$, so that $R_{k+1} = g(X_0) + Q$ Using the law of total variance, and noting that adding a constant does not affect variance, we have:

$$\begin{aligned} \text{var}(R_{k+1} \mid X_0 = i) &= \text{var}(Q \mid X_0 = i) \\ &= \text{var}(\mathbb{E}[Q \mid X_0 = i, X_1]) + \mathbb{E}[\text{var}(Q \mid X_0 = i, X_1)]. \end{aligned}$$

Let us consider the first term in the final sum above: The random variable $\mathbb{E}[Q \mid X_0 = i, X_1]$ takes the value $\mathbb{E}[Q \mid X_1 = j] = m_k(j)$ with probability p_{ij}. Given that $X_0 = i$, its mean is thus

$$\sum_j p_{ij} m_k(j) = m_{k+1}(i) - g(i),$$

and therefore its variance is

$$\sum_j p_{ij}(g(i) + m_k(j) - m_{k+1}(i))^2.$$

For the second term, notice that $\text{var}(Q \mid X_0 = i, X_1)$ is equal to $v_k(j)$ whenever X_1 happens to be j. Thus,

$$\mathbb{E}[\text{var}(Q \mid X_0 = i, X_1)] = \sum_j p_{ij} v_k(j).$$

Putting these two terms together, we find:

$$\text{var}(R_{k+1} \mid X_0 = i) = \sum_j p_{ij}(g(i)+m_k(j)-m_{k+1}(i))^2 + \sum_j p_{ij} v_k(j).$$

Exercise 8. Consider a discrete-time, finite-state Markov chain $\{X_t\}$, with states $\{1, \ldots, n\}$, and transition probabilities p_{ij}. States 1 and n are absorbing, that is, $p_{11} = 1$ and $p_{nn} = 1$. All other states are transient. Let A_1 be the event that the state eventually becomes 1. For any possible starting state i, let $a_i = \mathbf{P}(A_1 \mid X_0 = i)$ and assume that $a_i > 0$ for every $i \neq n$. Conditional on the information that event A_1 occurs, is the process X_n necessarily Markov? If yes, provide a proof, together with a formula for its transition probabilities. If not, provide a counterexample.

Solution: The answer is yes. Let B be an event of the form

$$B = \{X_0 = i_0, X_1 = i_1, \ldots, X_{t-1} = i_{t-1}\}.$$

It suffices to show that the transition probability $\mathbb{P}(X_{t+1} = j \mid X_t = i, A_1, B)$ is unaffected by the past history (the event B). We have

$$\mathbb{P}(X_{t+1} = j \mid X_t = i, A_1, B) = \frac{\mathbb{P}(X_{t+1} = j, A_1 \mid X_t = i, B)}{\mathbb{P}(A_1 \mid X_t = i, B)}.$$

By the Markov property of the process $\{X_t\}$ (the future is independent of the past, given the present), we have

$$\mathbb{P}(X_{t+1} = j, A_1 \mid X_t = i, B) = \mathbb{P}(X_{t+1} = j, A_1 \mid X_t = i),$$

and

$$\mathbb{P}(A_1 \mid X_t = i, B) = \mathbb{P}(A_1 \mid X_t = i),$$

from which the desired result follows.

Furthermore,

$$\begin{aligned}
\mathbb{P}(X_{t+1} = j \mid X_t = i, A_1) &= \frac{\mathbb{P}(X_{t+1} = j, A_1 \mid X_t = i)}{\mathbb{P}(A_1 \mid X_t = i)} \\
&= \frac{\mathbb{P}(A_1 \mid X_t = i, X_{t+1} = j)\mathbb{P}(X_{t+1} = j \mid X_t = i)}{\mathbb{P}(A_1 \mid X_t = i)} \\
&= \frac{p_{ij}a_j}{a_i}.
\end{aligned}$$

Exercise 9. A certain production device is in one of two states at any time: operational or repair. The operation time of the device is a random variable which is uniformly distributed over the integers $1, 2, \ldots, n$. The repair time has a deterministic value m which is a positive integer. The operation mode and repair mode alternate and the process continues indefinitely.

1. For every $1 \le i \le n$, and $t \ge 0$, let $X_t = i$ if the system is operational at time t and has been operational continuously for i time units (that is it was operational at times $t - i + 1, \ldots, t$, but was in the repair mode at time $t - i$). For every $n + 1 \le i \le n + m$, let $X_t = i$ if the system is in the repair mode at time t and the repair mode began at time $t - i + 1$. Show that X_t is a Markov chain. Identify the transition rates for this M.c., its transient and recurrent states and steady state distributions. How many are there?

2. Suppose we observe the system in steady state. What is the likelihood that at the time of the observation the system is operational and has been operational at least $(2/3)n$ time units?

Solution:

1. The Markov chain has been drawn in Figure 1. We can justify the transition probabilities out of state i for $1 \le i \le n$ as follows. If $X \sim$ Uni$(\{1, \ldots, n\})$, then X conditional on $X \ge i$ is Uni$(\{i, i + 1, \ldots, n\})$.

A simple computation shows this. Thus as $|\{i, \dots, n\}| = n - i + 1$, with probability $1/(n-i+1)$ the machine will fail in the next time period. As the dynamics of X_n are described by the one step transition probabilities from each state, X_n is a Markov chain. As there is the path $1, 2, \dots, n, n+1, \dots, n+m, 1$, all the states are in the same recurrent class.

If $n = 1$, the MC is periodic, and not very interesting. From now on, we assume $n > 1$. Then the MC is aperiodic (this takes some work, but intuitively, if we can have loops of size $m+1$ or $m+2$, then for all $n > (m+2)^2$, you can be in state one). From the steady state equation $\pi' P = \pi'$, we have

$$\begin{aligned}
\pi_1 &= \pi_{n+m} \\
\pi_2 &= \left(\frac{n-1}{n}\right)\pi_1 \\
\pi_3 &= \left(\frac{n-2}{n-1}\right)\pi_2 = \frac{n-2}{n}\pi_1 \\
&\vdots \\
\pi_i &= \left(\frac{n-i+1}{n-i+2}\right)\pi_{i-1} = \frac{n-i+1}{n}\pi_1 && i = 2, \dots, n \\
\pi_{n+1} &= \frac{1}{n}\pi_1 + \frac{1}{n-1}\pi_2 + \cdots + \frac{1}{2}\pi_{n-1} + \pi_n \\
\pi_{n+2} &= \pi_{n+1} \\
&\vdots \\
\pi_{n+i} &= \pi_{n+i-1} && i = 2, \dots, m
\end{aligned}$$

We can ignore the equation for π_{n+1} as we have a redundant equation, and we note that from the final $m-2$ equations and the first equation, we have that $\pi_1 = \pi_{n+i}$ for all $i = 1, \dots, m$. Now using that the probabilities must sum to one, we have

$$1 = \pi_1 + \sum_{i=2}^{n} \frac{n-i+1}{n}\pi_1 + m\pi_1 = \pi_1\left(m + 1 + \frac{n-1}{2}\right),$$

thus for $i = 1, \dots, m$,

$$\pi_1 = \frac{1}{m+1+\frac{n-1}{2}} = \pi_{n+i}.$$

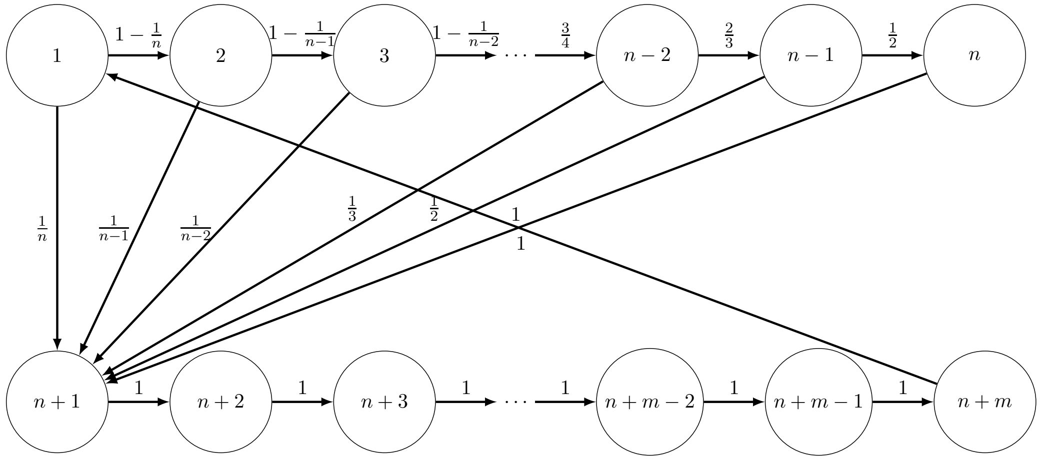

Figure 1: The Markov chain for problem 11. In the top row, the machine is in operation, while in the bottom row, the machine is undergoing repair.

and for $i = 2, \dots, n$,

$$\pi_i = \frac{1}{m + 1 + \frac{n-1}{2}} \frac{n - i + 1}{n}$$

2.

$$\mathbb{P}\left(X_\infty \in \left\{\frac{2n}{3}, \dots, n\right\}\right) = \sum_{i=2n/3}^{n} \pi_i = \pi_1 \sum_{i=2n/3}^{n} \frac{n-i+1}{n} = \pi_1\left(\frac{n}{18} + \frac{1}{n} + \frac{1}{2}\right)$$