Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: Discrete-Time Signal Processing

Fall 2005

Solutions for Problem Set 6

Issued: Tuesday, October 25 2005.

Problem 6.1

OSB Problem 4.52

(a) We use the polyphase implementation discussed in OSB pages 182–184.

$$\begin{aligned} h_1[n] &= a\delta[n] + c\delta[n-1] + e\delta[n-2] \\ h_2[n] &= b\delta[n] + d\delta[n-1] \\ h_3[n] &= \delta[n-1] \end{aligned}$$

(b) For $h[n]$, 5 multiplications are required for each sample of $y_1[n]$.

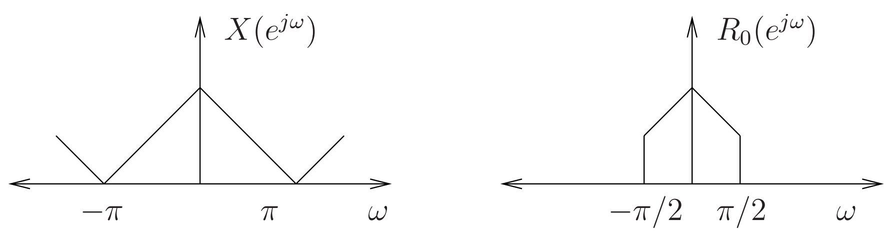
For $h_1[n]$, 1.5 multiplications are required for each sample of $w_1[n]$. For $h_2[n]$, 1 multiplication is required for each sample of $w_2[n]$. If $h_3[n]$ is implemented as an FIR filter, then 2 multiplications are required for each sample of $w_3[n]$. In total, 4.5 multiplications are required for each sample of $y_2[n]$.

Note that if $h_3[n]$ were built into the flowgraph as a delay, then 2.5 multiplications would be required for each sample of $y_2[n]$.

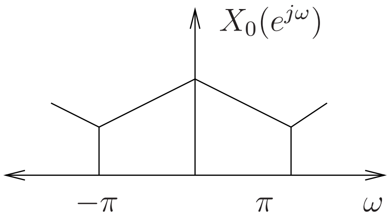
Problem 6.2

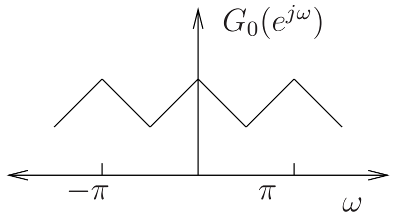
OSB Problem 4.53

(a) First, we plot $X(e^{j\omega})$ (left plot below), and $R_0(e^{j\omega})$, after low-pass filtering with cut-off at $\pi/2$ (right plot below):



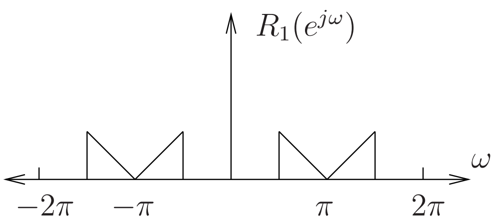
The downsampler expands the frequency axis. Since $R_0(e^{j\omega})$ is bandlimited to $\pi/2$, no aliasing occurs (left plot below). The upsampler compresses the frequency axis by a factor of 2 (right plot below).



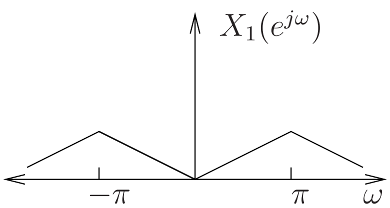


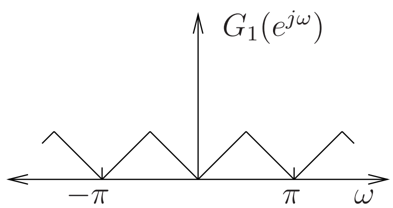
The lowpass filter cuts off at $\pi/2$, hence $Y_0(e^{j\omega}) = R_0(e^{j\omega})$ as sketched above.

Although not a formal requirement for this problem, additional insight can be gained by considering the bottom branch of the system. $R_1(e^{j\omega})$ appears in the plot below.



$R_1(e^{j\omega})$ is not bandlimited to $\pi/2$, so aliasing appears in $X_1(e^{j\omega})$ after expanding the frequency axis due to downsampling. However, since $R_1(e^{j\omega})$ is zero from $-\pi/2$ to $\pi/2$, the aliased component does not distort the original one (left plot below). The upsampler compresses the frequency axis by a factor of 2 (right plot below).





The highpass filter cuts off at $\pi/2$, hence $Y_1(e^{j\omega}) = R_1(e^{j\omega})$ as sketched above.

(b)

$$
\begin{aligned}
R_0(e^{j\omega}) &= H_0(e^{j\omega})X(e^{j\omega}) \\
X_0(e^{j\omega}) &= 0.5R_0(e^{j\frac{\omega}{2}}) + 0.5R_0(e^{j(\frac{\omega}{2}+\pi)}) \\
G_0(e^{j\omega}) &= X_0(e^{j2\omega}) \\
&= 0.5R_0(e^{j\omega}) + 0.5R_0(e^{j(\omega+\pi)}) \\
&= 0.5H_0(e^{j\omega})X(e^{j\omega}) + 0.5H_0(e^{j(\omega+\pi)})X(e^{j(\omega+\pi)}) \\
Y_0(e^{j\omega}) &= G_0(e^{j\omega})H_0(e^{j\omega}) \\
&= 0.5H_0^2(e^{j\omega})X(e^{j\omega}) + 0.5H_0(e^{j\omega})H_0(e^{j(\omega+\pi)})X(e^{j(\omega+\pi)})
\end{aligned}
$$

(c) Similar to part (b), we have:

$$
Y_1(e^{j\omega}) = 0.5H_1^2(e^{j\omega})X(e^{j\omega}) + 0.5H_1(e^{j\omega})H_1(e^{j(\omega+\pi)})X(e^{j(\omega+\pi)})
$$

Since $H_1(e^{j\omega}) = H_0(e^{j(\omega+\pi)})$,

$$
\begin{aligned}
Y_0(e^{j\omega}) &= 0.5H_0^2(e^{j\omega})X(e^{j\omega}) + 0.5H_1(e^{j\omega})H_0(e^{j(\omega)})X(e^{j(\omega+\pi)}) \\
Y_1(e^{j\omega}) &= 0.5H_1^2(e^{j\omega})X(e^{j\omega}) + 0.5H_1(e^{j\omega})H_0(e^{j(\omega)})X(e^{j(\omega+\pi)}) \\
Y(e^{j\omega}) &= Y_0(e^{j\omega}) + Y_1(e^{j\omega}) \\
&= 0.5[H_1^2(e^{j\omega}) + H_0^2(e^{j\omega})]X(e^{j\omega}) + H_1(e^{j\omega})H_0(e^{j(\omega)})X(e^{j(\omega+\pi)})
\end{aligned}
$$

Thus, the general condition to guarantee that $y[n]$ is proportional to $x[n-n_d]$ for any stable input $x[n]$:

$$
\begin{aligned}
H_1^2(e^{j\omega}) + H_0^2(e^{j\omega}) &= Ke^{-jn_d\omega} \\
H_1(e^{j\omega})H_0(e^{j(\omega)}) &= 0
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
H_0^2(e^{j(\omega+\pi)}) + H_0^2(e^{j\omega}) &= Ke^{-jn_d\omega} \\
H_0(e^{j(\omega+\pi)})H_0(e^{j(\omega)}) &= 0
\end{aligned}
$$

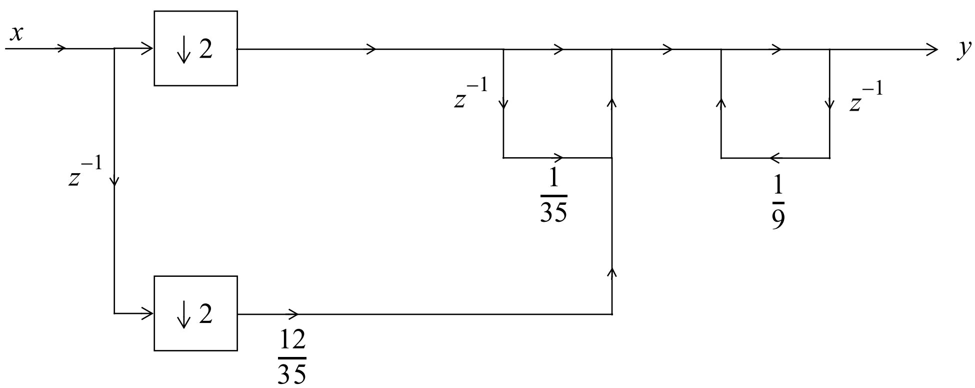
Problem 6.3

(a) We need 4 multiplications for each value before the downsampler and 8 multiplications on average for each output sample of y.

(b) We can write the system transfer function:

$$H(z) = \frac{\left(1+\frac{1}{5}z^{-1}\right)\left(1+\frac{1}{7}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)} = \frac{1+\frac{12}{35}z^{-1}+\frac{1}{35}z^{-2}}{1-\frac{1}{9}z^{-2}}$$

This can be implemented using the flowgraph shown below:



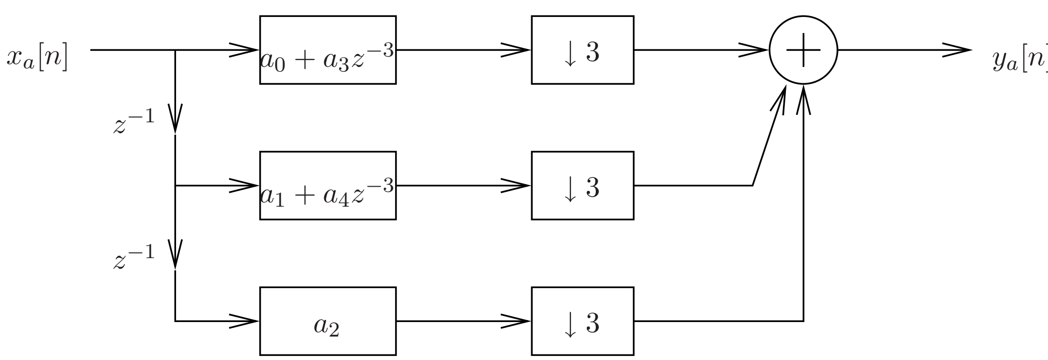
(c) Now the whole computational requirement is only 3 multiplications on average for each output sample of y.

Problem 6.4

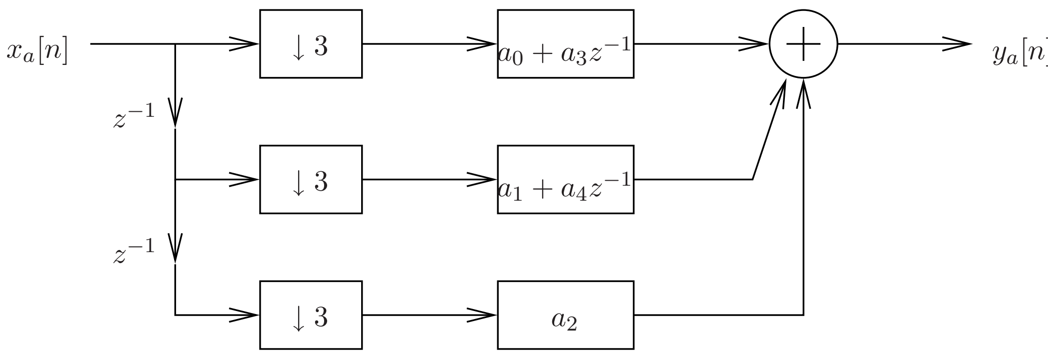
(Problem 4, Spring 2005 Midterm)

(a) This part is a straightforward application of the polyphase decomposition. The number of polyphase components should be a multiple of 3 to be able to take advantage of the downsampling.

The polyphase decomposition and swapping the summer with the downsampling yields

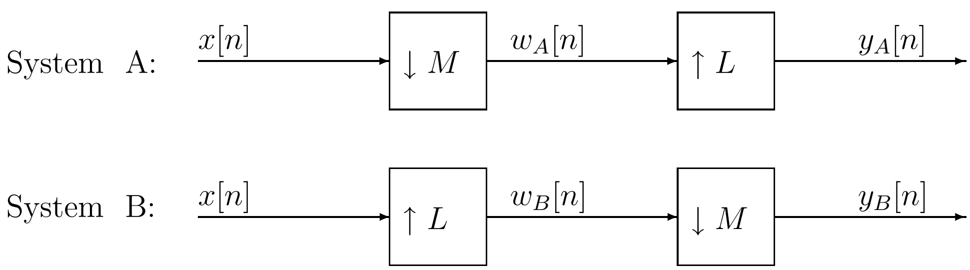


Now applying the downsampling noble identity yields



This is efficient because the filtering is done at the lowest possible sampling rate.

(b) $\uparrow 3$ and $\downarrow 2$ can be swapped since 3 and 2 are coprime. This result can seen by comparing the following two systems:



The following equations describe the stages of System A:

$$w_A[n] = x[2n]$$

$$y_A[n] = \begin{cases} w_A[\frac{n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

The following equations describe the stages of System B:

$$w_B[n] = \begin{cases} x[\frac{n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

$$y_B[n] = w_B[2n]$$

Therefore,

$$y_A[n] = \begin{cases} x[\frac{2n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

and

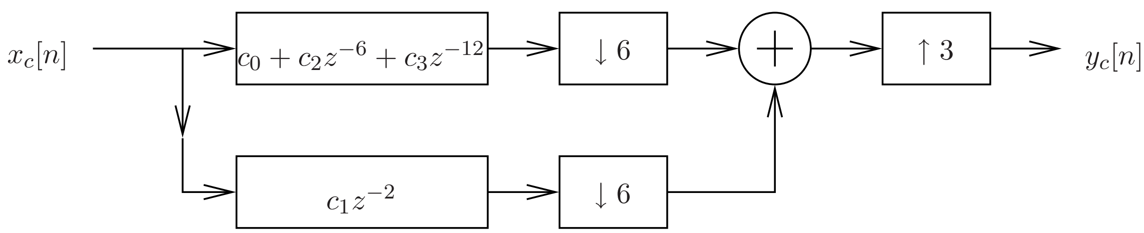
$$y_B[n] = \begin{cases} x[\frac{2n}{3}] & \text{if } \frac{2n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Because for all integer values of n for which $\frac{n}{3}$ is an integer, $\frac{2n}{3}$ is also an integer and vice-versa, the systems are equivalent and can be swapped.

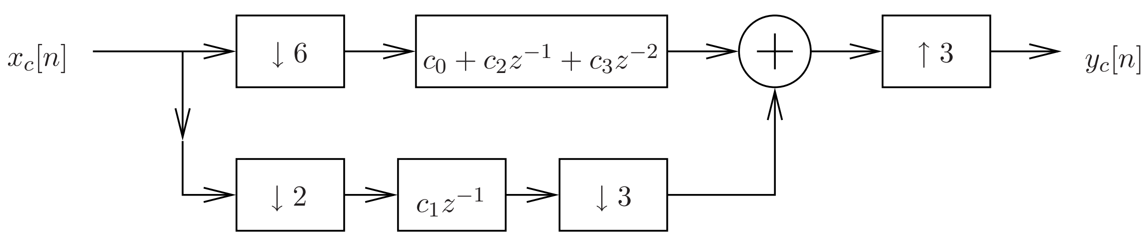
This swap therefore leaves $H_b(z)$, which can be written in terms of z^6, in front of a downsampling by 6. An efficient implementation is then obtained by applying the downsampling noble identity:

$$x_b[n] \rightarrow \boxed{\downarrow 6} \rightarrow \boxed{b_0 + b_1 z^{-1} + b_2 z^{-2}} \rightarrow \boxed{\uparrow 3} \rightarrow y_b[n]$$

(c) Here it is easy to say, "I can *almost* write $H_c(z)$ as a function of z^6." To take advantage of this near miss, implement $H_c(z)$ as $c_0 + c_2 z^{-6} + c_3 z^{-12}$ in parallel combination with $c_1 z^{-2}$:



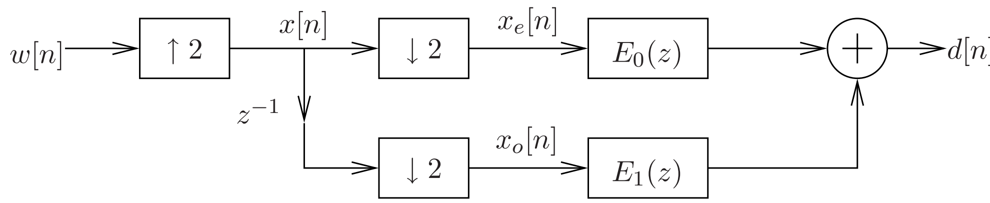
Now application of downsampling identities gives:



Note also that this answer can be obtained in a perfectly systematic way by applying a polyphase decomposition. It turns out that some polyphase filters are zero.

Problem 6.5

(a) After a polyphase decomposition of $h[n]$ one obtains the block diagram below:



Now since $x_o[n] = 0$ for all n and $x_e[n] = w[n]$ for all n, $d[n]$ is the output of an LTI filter $E_0(z)$ driven by input $w[n]$, where $E_0(z)$ is the polyphase component containing only even values of $h[n]$, *i.e.* $e_0[n] = h[2n]$. This gives us simply:

$$w[n] \rightarrow \boxed{h[2n]} \rightarrow d[n]$$

(b) Since upon simplification it becomes evident that the system is LTI, we know that the output must be WSS for the input random sequence specified. Because of this, $R_{dd}[n, m]$ has no dependence on n and is simply

$$R_{dd}[n, m] = \sigma_w^2 \sum_{k=-\infty}^{\infty} e_0[k]e_0[m + k] = \sigma_w^2 \sum_{k=-\infty}^{\infty} h[2k]h[2(m + k)] \ ,$$

the convolution of $\sigma_w^2 \delta[p]$ with $h[2p] * h[-2p]$ for integer p.

Problem 6.6

(a) There is one output sample generated for every pair of input samples. Even input samples require 3 multiplies and odd input samples require 2 multiplies. Thus each pair requires 5 multiplies.

(b) Applying the compressor identity to the previous structure results in:

$$H(z) = H_0(z^2) + z^{-1}H_1(z^2).$$

From the difference equations in the previous part we have:

$$H_0(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-1}}{1 - \frac{1}{4}z^{-1}},$$

and

$$H_1(z) = \frac{\frac{1}{12}}{1 - \frac{1}{4}z^{-1}}.$$

Thus,

$$H(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-2} + \frac{1}{12}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{-\frac{1}{3}(1 - \frac{3}{4}z^{-1})(1 + \frac{1}{2}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} = \frac{-\frac{1}{3} + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}.$$

Therefore, $a = 1/2, b = -1/3$ and $c = 1/4$.

(c) In this implementation 3 multiplies are required for every input sample. For every output sample we need to calculate 2 values of $v[n]$. Altogether we need 6 multiplies per output sample.

Problem 6.7

(a)

$$S(z) = \frac{2}{1 - \frac{1}{3}z^{-1}} + \frac{3}{1 + \frac{1}{2}z^{-1}} = \frac{5}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}.$$

Thus, $a_1 = -1/6, a_2 = 1/6$.

(b) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{2} a_k \phi_s[i-k], \quad i = 1, 2,$$

or, in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] \\ \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \end{bmatrix}.$$

(c) Denote $s_1[n] = 2(\frac{1}{3})^n u[n]$, $s_2[n] = 3(-\frac{1}{2})^n u[n]$. Then for $m > 0$,

$$\phi_{s_1}[m] = \frac{9}{2}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_2}[m] = 12\left(-\frac{1}{2}\right)^m$$

$$\phi_{s_{12}}[m] = \frac{36}{7}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_{21}}[m] = \frac{36}{7}\left(-\frac{1}{2}\right)^m.$$

Thus,

$$\phi_s[m] = \frac{135}{14}\left(\frac{1}{3}\right)^{|m|} + \frac{120}{7}\left(-\frac{1}{2}\right)^{|m|}.$$

So, $\phi_s[0] = 26.78$, $\phi_s[1] = -5.36$ and $\phi_s[2] = 5.36$.

(d) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6$.

(e) These values are the same as those we found in part (a), as expected.

(f) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{3} a_k \phi_s[i-k], \quad i = 1, 2, 3,$$

or, in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] & \phi_s[2] \\ \phi_s[1] & \phi_s[0] & \phi_s[1] \\ \phi_s[2] & \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \\ \phi_s[3] \end{bmatrix}.$$

(g) $\phi_s[3] = -1.79$.

(h) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6, a_3 = 0$.

(i) The signal $s[n]$ is the impulse response of an all-pole filter with two poles, i.e. second order. Therefore, $a_k = 0$ for $k > 2$.

(j) No, since the signal corresponds to the impulse response of a second order filter. The higher order coefficients will all be 0.