Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: Discrete-Time Signal Processing

Fall 2005

Solutions for Problem Set 2

Issued: Tuesday, September 20 2005.

Problem 2.1

(i) Real-valued impulse response:

Poles that aren't real must be in complex conjugate pairs. Zeros that aren't real must be in complex conjugate pairs.

(ii) Finite impulse response:

All poles are at the origin. The ROC is the entire z-plane, except possibly $z = 0$.

(iii) $h[n] = h[2\alpha - n]$ where 2α is an integer:

Causality combined with the given symmetry property implies a finite-length $h[n]$ that can only be nonzero between time zero and time 2α. Thus we must have all poles at the origin and at most 2α zeros. The z transform of $h[2\alpha - n]$ is $z^{-2\alpha}H(1/z)$, so any zero of $H(z)$ at $c \neq 0$ must be paired with a zero at $1/c$.

(iii) Minimum phase:

All poles and zeros are inside the unit circle (so that the inverse can be stable and causal).

(iv) All-pass:

Each pole is paired with a zero at the conjugate reciprocal location.

Problem 2.2

(a) False. As a counterexample, consider a filter described by

$$H(e^{j\omega}) = 1 - \frac{1}{2}e^{-j\omega}.$$

We know that $H(z)$ can describe a causal, stable filter since its single pole is at $z = 0$. In the vicinity of $\omega = 0$, $\theta(\omega)$ is increasing, so the group delay $\tau(\omega)$ is negative.
The group delay evaluated at $\omega = 0$ for this example can be determined using equation (5.67) in OSB:

$$grd[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega - \theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}$$

and substituting $r = \frac{1}{2}$ and $\theta = 0$.

(b) False. Any zero-phase FIR filter may be delayed to become causal, and the resulting filter will have the same phase as the delay block which was applied to it. One counterexample is the causal, stable filter described by $H(z) = 1 + 2z^{-1} + z^{-2}$.

(c) True. Note that

$$\int_0^{\pi} \tau(\omega)d\omega = -\left.\theta(\omega)\right|_{\omega=0}^{\pi} = \theta(0) - \theta(\pi).$$

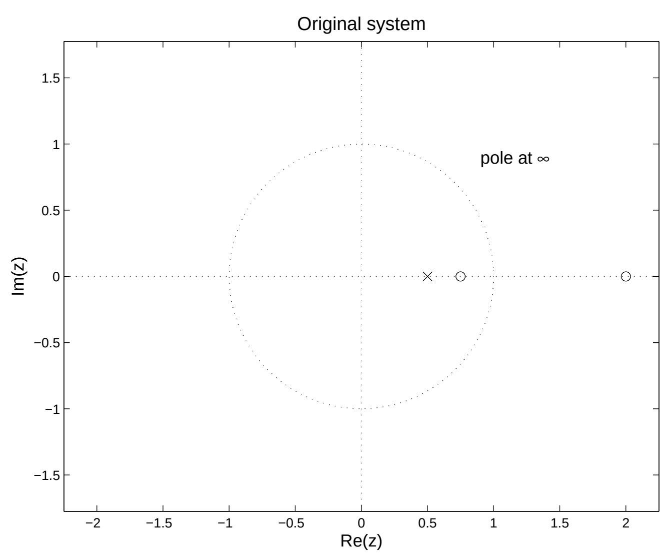
Setting this to 0, we realize that the assertion is true if having minimum phase and all poles and zeros on the real axis implies

$$\theta(0) = \theta(\pi).$$

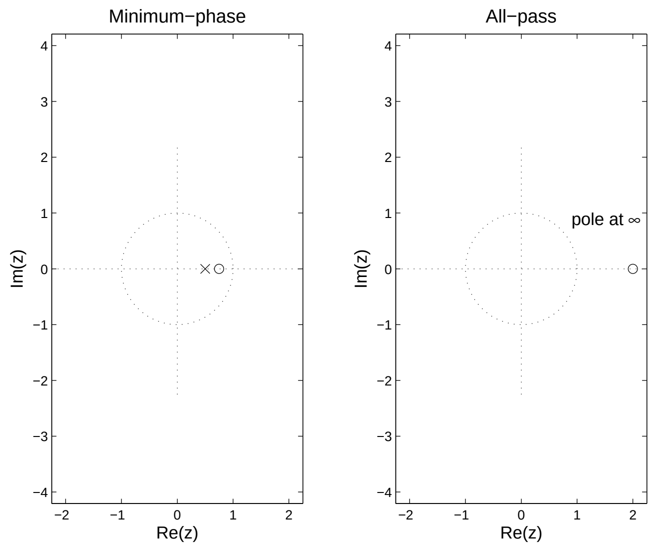
A minimum phase filter with poles and zeros on the real axis will have all poles and zeros between -1 and 1. Factoring $H(e^{j\omega})$ to find poles and zeros, we notice that for a given pole term $\frac{1}{1-ae^{-j\omega}}$, the restriction $-1 < a < 1$ yields a phase contribution of 0 when $\omega = 0$ or $\omega = \pi$. Likewise, a given zero term $1 - be^{-j\omega}$ gives a phase contribution of 0 when $\omega = 0$ or $\omega = \pi$ as long as $-1 < b < 1$. Since all poles and zeros of such a minimum phase filter meet these criteria, $\theta(0) = \theta(\pi) = 0$ Therefore, $\int_0^{\pi} \tau(\omega)d\omega = 0$. (Note that $\int_0^{\pi} \tau(\omega)d\omega = 0$ for all real minimum phase filters, even if there are complex poles and zeros.)

Problem 2.3

The pole-zero diagram for the original system is as follows:

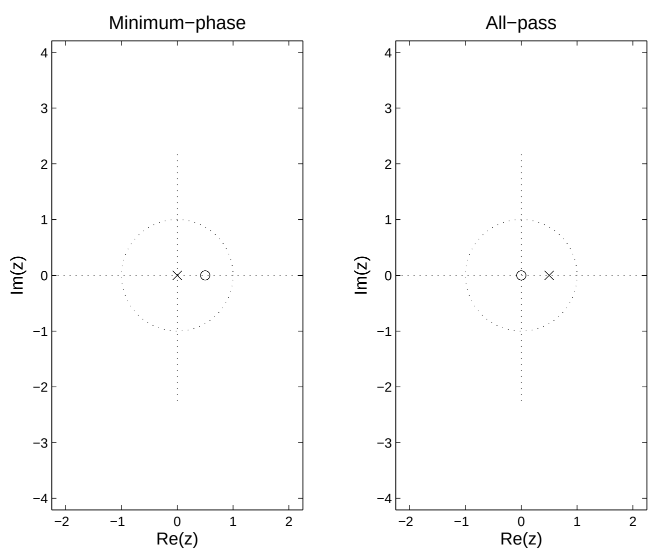


(a) One way to carry out the minimum-phase and all-pass decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the all-pass system.

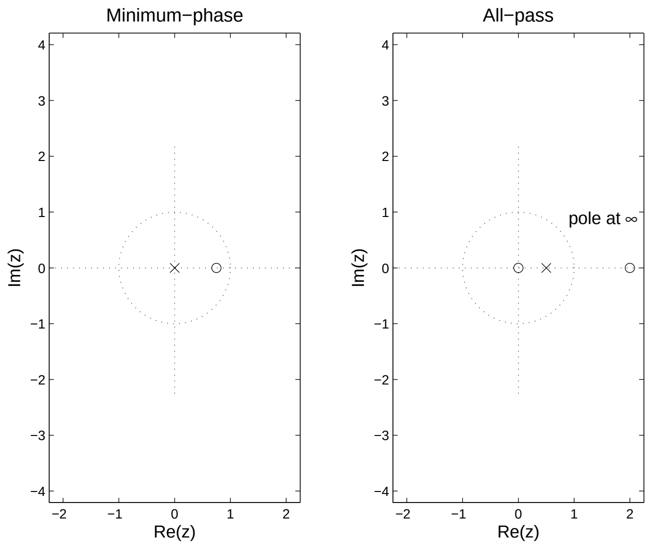


Next, we need to modify the first stage because we need to make sure that the all-pass system really is all-pass, so add a pole at $z = 1/2$ and a zero at $z = 0$.

To preserve the original system, we can cancel these newcomers by placing a zero at $z = 1/2$ and a pole at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and all-pass systems are as shown below.



In the minimum-phase system, the pole at $z = 1/2$ from the first stage has been cancelled by the zero added in the second stage. Another way to look at that is that for this particular system, we started with an all-pass pair (a pole at $z = 1/2$ and a zero at $z = 2$, so we could have put these into the all-pass system initially.

The minimum-phase system function is:

$$\begin{aligned} H_{M1}(z) &= \frac{z - \frac{3}{4}}{z} \\ &= 1 - \frac{3}{4} z^{-1} \end{aligned}$$

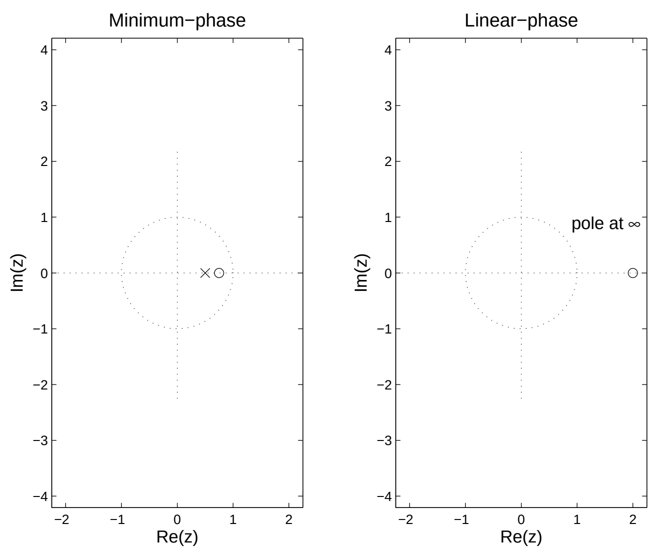
The all-pass system function is:

$$\begin{aligned} H_{ap}(z) &= \frac{z(z-2)}{z - \frac{1}{2}} \\ &= \frac{1 - 2z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right) z^{-1}} \end{aligned}$$

In constructing these systems, we didn't come across any decision where we could have chosen different routes. If we wanted to change one of the systems, we would have to add the same number of poles and zeros to it, and these would have to be cancelled by zeros and poles in the other system to preserve the original system.

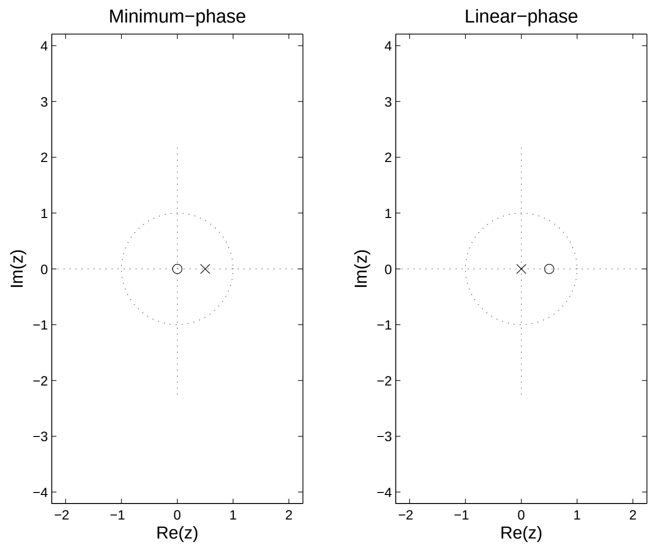
We can't add poles or zeros to the minimum phase system, because if we did, then when we added the cancelling zeros or poles to the all-pass system, they would have to be reflected outside the unit circle to keep the latter system all-pass. These items outside the unit circle could not be cancelled in the minimum phase system. Finally, we cannot change the all-pass system because if we added a zero and a pole, then to keep the system all-pass, we would have to reflect a pole or zero to the other side of the unit circle, and the items outside the unit circle could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique up to a scale factor.

(b) One way to carry out the minimum-phase and FIR linear-phase decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the linear-phase system.

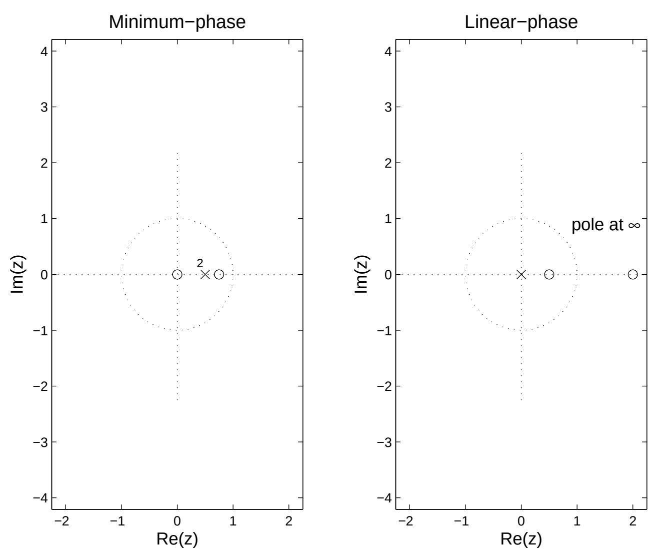


Next, we need to modify the first stage because we need to make sure that the linear-phase FIR system really is linear-phase FIR, so add a zero at $z = 1/2$. Since the system has to have the same number of zeros and poles, we also need to add a pole. For an FIR system, the pole must be at $z = 0$ or at $z = \infty$. We choose to add the pole at $z = 0$ because we will have to cancel the pole by a zero in the minimum-phase system.

To preserve the original system, we can cancel these newcomers by placing a pole at $z = 1/2$ and a zero at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and FIR linear-phase systems are as shown below.



The minimum phase system function is:

$$
\begin{aligned}
H_{M2}(z) &= \frac{z\left(z-\frac{3}{4}\right)}{\left(z-\frac{1}{2}\right)^2} \\
&= \frac{1-\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}
\end{aligned}
$$

The FIR generalized linear-phase system function is:

$$
\begin{aligned}
H_L(z) &= \frac{\left(z-\frac{1}{2}\right)(z-2)}{z} \\
&= z\left[\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\right] \\
&= z-2.5+z^{-1}
\end{aligned}
$$

Since this expression for $H_L(z)$ has even symmetry and an odd number of taps, we would not necessarily expect a zero at $z=1$ or at $z=-1$, and this is consistent with the pole-zero diagram above. In constructing these systems, we didn't come across any decisions where we could have chosen different routes. Furthermore, we cannot change the minimum phase system. If we tried adding a pole and zero to it, these would have to be cancelled in the FIR linear phase system. But the zero in the linear-phase system would have to be reflected outside the unit circle to maintain linear-phase, and this could not be compensated for in the minimum-phase system. Similarly, we cannot add a pole and zero to the FIR linear-phase system because if we did, then to keep it linear-phase, we would have to reflect the zero outside the unit circle, and this could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

Problem 2.4

a. We desire $|H(z)H_c(z)|=1$, where $H_c(z)$ is stable and causal and $H(z)$ is not minimum phase. So,

$$|H_{ap}(z)H_{min}(z)H_c(z)|=1$$

Since $|H_{ap}(z)|=1$, we want

$$|H_{min}(z)H_c(z)|=1$$

This means we have

$$H_c(z)=\frac{1}{H_{min}(z)}$$

which will be stable and causal since all the zeros of $H_{min}(z)$, which become the poles of $H_c(z)$, are inside the unit circle.

b. Since

$$H_c(z)=\frac{1}{H_{min}(z)}$$

We have

$$G(z)=H_{ap}(z)$$

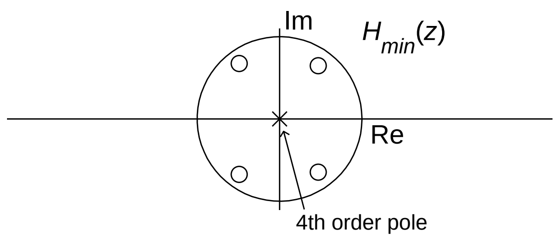
c.

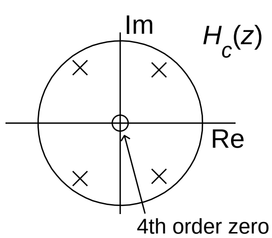
$$H(z) = (1 - 0.8e^{j0.3\pi}z^{-1})(1 - 0.8e^{-j0.3\pi}z^{-1})(1 - 1.2e^{j0.7\pi}z^{-1})(1 - 1.2e^{-j0.7\pi}z^{-1})$$

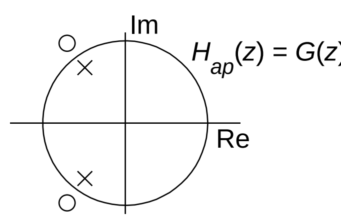
$$H_{min}(z) = (1.44)(1-0.8e^{j0.3\pi}z^{-1})(1-0.8e^{-j0.3\pi}z^{-1})(1-(5/6)e^{j0.7\pi}z^{-1})(1-(5/6)e^{-j0.7\pi}z^{-1})$$

$$H_c(z) = \frac{1}{(1.44)(1 - 0.8e^{j0.3\pi}z^{-1})(1 - 0.8e^{-j0.3\pi}z^{-1})(1 - (5/6)e^{j0.7\pi}z^{-1})(1 - (5/6)e^{-j0.7\pi}z^{-1})}$$

$$G(z) = H_{ap}(z) = \frac{(z^{-1} - (5/6)e^{-j0.7\pi})(z^{-1} - (5/6)e^{j0.7\pi})}{(1 - (5/6)e^{j0.7\pi}z^{-1})(1 - (5/6)e^{-j0.7\pi}z^{-1})}$$







Problem 2.5

(a) Using the all-pass principle, $H(z)$ can be represented as a cascade of the minimum phase term $H_{min}(z)$ and an all-pass term. We add a pole to remove the zero at $z = z_k$, and replace it with a zero at $z = \frac{1}{z_k^*}$:

$$H(z) = H_{min}(z)\frac{z^{-1} - z_k^*}{1 - z_k z^{-1}} = Q(z)(z^{-1} - z_k^*)$$

(b) We know that:

$$H(z) = Q(z)(z^{-1} - z_k^*) = Q(z)z^{-1} - Q(z)z_k^*$$

$$H_{min}(z) = Q(z)(1 - z_k z^{-1}) = Q(z) - z_k z^{-1}Q(z)$$

Taking the inverse z-transform:

$$h[n] = q[n-1] - z_k^* q[n]$$

$$h_{min}[n] = q[n] - z_k q[n-1]$$

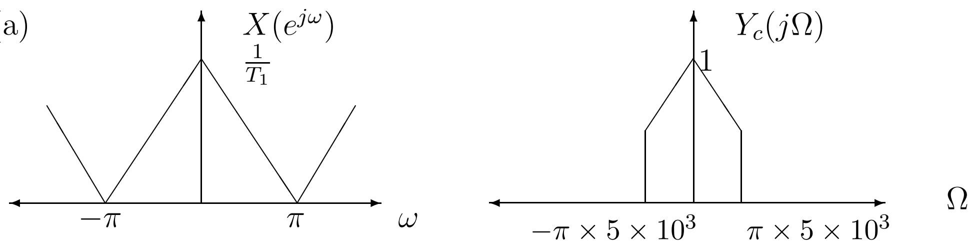
(c) Using our answer from part (b):

$$\begin{aligned}
\varepsilon &= \sum_{n=0}^{m} |h_{min}[n]|^2 - \sum_{n=0}^{m} |h[n]|^2 \\
&= \sum_{n=0}^{m} |q[n]|^2 - z_k q[n-1] q^*[n] - z_k^* q^*[n-1] q[n] + |z_k|^2 |q[n-1]|^2 \\
&\quad - \sum_{n=0}^{m} |q[n-1]|^2 - z_k q[n-1] q^*[n] - z_k^* q^*[n-1] q[n] + |z_k|^2 |q[n]|^2 \\
&= \sum_{n=0}^{m} (1 - |z_k|^2) |q[n]|^2 - \sum_{n=0}^{m} (1 - |z_k|^2) |q[n-1]|^2 \\
&= (1 - |z_k|^2) |q[m]|^2
\end{aligned}$$

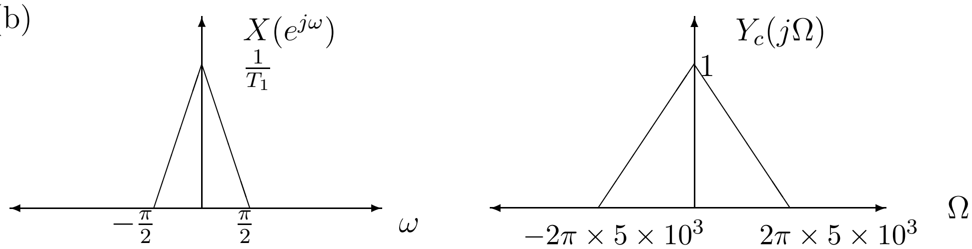
(d) Since $|z_k| < 1$, $(1 - |z_k|^2)$ is positive and therefore $(1 - |z_k|^2)|q[m]|^2 > 0$
Then $\sum_{n=0}^{m} |h_{min}[n]|^2 - \sum_{n=0}^{m} |h[n]|^2 > 0$ thus $\sum_{n=0}^{m} |h_{min}[n]|^2 > \sum_{n=0}^{m} |h[n]|^2$

Problem 2.6

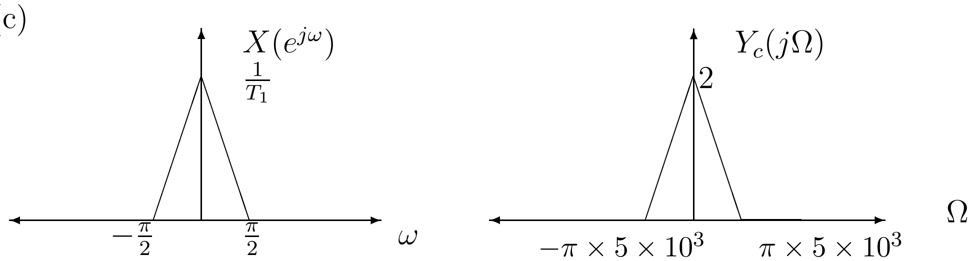
(a)



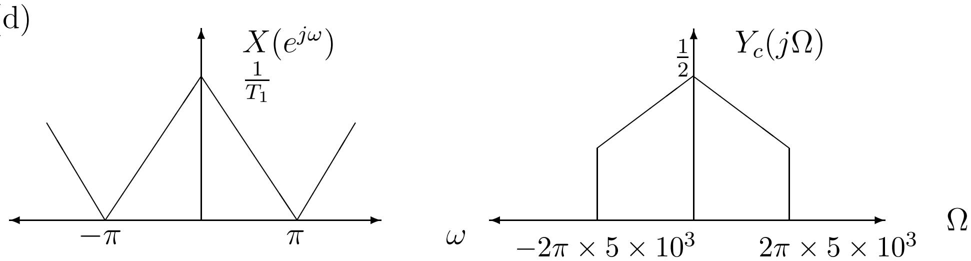
(b)



(c)

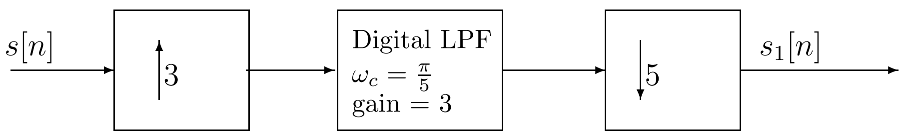


(d)



Problem 2.7

In both systems, the speech was filtered first so that the subsequent sampling results in no aliasing. Therefore, going from $s[n]$ to $s_1[n]$ basically requires changing the sampling rate by a factor of $3kHz/5kHz = 3/5$. This is done with the following system:

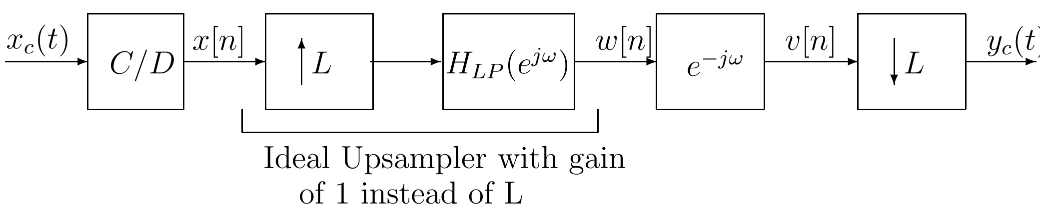


Problem 2.8

Split $H(e^{j\omega})$ into a lowpass and a delay.

$$H(e^{j\omega}) = H_{LP}(e^{j\omega})e^{-j\omega}$$

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{L} \\ 0, & \frac{\pi}{L} < |\omega| \leq \pi \end{cases}$$



Then we analyze the system as follows:

$$x[n] = x_c(nT) \quad \text{no aliasing assumed}$$

$$w[n] = \frac{1}{L}x_c(n\frac{T}{L}) \quad \text{rate change}$$

$$v[n] = w[n-1] = \frac{1}{L}x_c\left(n\frac{T}{L} - \frac{T}{L}\right), \quad \text{delay at higher rate}$$

$$y[n] = v[nL] = \frac{1}{L}x_c\left(nT - \frac{T}{L}\right)$$

Problem 2.9

(a) Let's rewrite System A as the cascade of two systems: an S/I (sample to impulse) converter followed by a CT filter. The S/I converter turns the DT signal $x_d[n]$ into a CT impulse train. If we allow the output of the S/I converter to be $x_s(t)$, then we have

$$x_s(t) = \sum_{k=-\infty}^{\infty} x_d[k]\delta(t - kT_1)$$

Then, the output $y_c(t)$ of the CT filter is the convolution of $x_s(t)$ and $h_1(t)$, or

$$y_c(t) = x_s(t) * h_1(t).$$

We see that by combining the above two equations, we get the equation that describes the behavior of System A.

$x_c(t)$ is bandlimited to $\Omega_c = \pi \cdot 10^{-3}$ rad/sec. Thus, we know that we can guarantee the equality of $x_c(t)$ and $y_c(t)$ when T is sufficiently small (*i.e.* no aliasing from the first C/D converter) and System A is an ideal D/C converter with the same sampling period.

We have no aliasing when $\Omega_c T < \pi$, or when $T < 1000$. System A is an ideal D/C converter when $h_1(t)$ is an appropriate sinc function.

Thus, the following conditions work:

$$\begin{aligned} T &= 500. \\ T_1 &= 500. \\ h_1(t) &= \frac{\sin(\pi t/T)}{\pi t/T} \end{aligned}$$

(b) As we saw in the solution to part (a), our choices are not unique. We can choose any T such that $T < 1000$. However, we see that we need $T_1 = T$. We also have a choice regarding $h_1(t)$. Since $X_d(e^{j\omega})$, or the Fourier transform of $x_d[n]$, is zero for

$$\frac{T\pi}{1000} < |\omega| < \pi,$$

$X_s(j\Omega)$, or the Fourier transform of $x_s(t)$, is zero for

$$\frac{\pi T}{1000 T_1} = \frac{\pi}{1000} < |\Omega| < \frac{\pi}{T_1}.$$

Thus, $H_1(j\Omega)$, or the Fourier transform of $h_1(t)$, can be anything in that frequency range (and, by extension, any "copy" of this section of the frequency spectrum). If it is a constant of T for $|\Omega| < \frac{\pi}{1000}$ and zero for $|\Omega| > \frac{\pi}{T}$, then we have $y_c(t) = x_c(t)$.

(c) Since we are interested only in the operations between $x_d[n]$ and $y_d[n]$, we need not worry about aliasing from the first C/D converter in the whole system destroying our hopes for achieving consistent resampling. Thus, there are no absolute restrictions on T and T_1 like we had in parts (a) and (b); they may, however, be related to each other.

In other words, what is going on between $x_d[n]$ and $y_d[n]$? System A is taking each sample of $x_d[n]$ and replacing it with $h_1(t)$ delayed by nT_1 and scaled by $x_d[n]$ at that point. Then, the C/D converter resamples the result.

Let's consider what happens with $x_d[n] = \delta[n - n_0]$ for an integer n_0. Then, $y_c(t) = h_1(t - n_0 T_1)$. The sampled version of $y_c(t)$ is

$$\begin{aligned} y_d[n] &= y_c(nT) \\ &= h_1(nT - n_0 T_1). \end{aligned}$$

A condition for consistent resampling is thus

$$h_1(nT - n_0 T_1) = \delta[n - n_0].$$

Because of the linearity of the mapping from $x_d[n]$ to $y_d[n]$, this is actually the only condition that must be checked. To simplify the condition further, we have

$$\text{evaluating at } n = n_0\text{:} \quad 1 = h_1(n(T - T_1))$$

$$\text{evaluating at } n \neq n_0\text{:} \quad 0 = h_1(nT - n_0 T_1))$$

The case of practical significance is to have $T = T_1$, in which case we find that $h_1(t)$ should satisfying an *interpolating condition*: $h_1(0) = 1$ and $h_1(t) = 0$ for all multiples of T. (It doesn't matter what $h_1(t)$ is for other values of t.)