Lectures on Dynamic Systems and Control

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Chapter 25

Minimal State-Space Realization

25.1 Introduction

Our goal in this lecture and a couple that follow is to further explore the "structural" significance of the assumptions of reachability and observability, and to understand their role in connecting the input/output (or transfer function) description of a system to its internal (or state-space) description. The development will be phrased in the language of DT systems, but the results hold unchanged (apart from some details of interpretation) for the CT case.

25.2 The Kalman Decomposition

In earlier lectures we presented two types of standard forms, one that depended on a separation of the state space into the reachable subspace and its complement, and another that separated the state space into the unobservable subspace and its complement. The question naturally arises as to whether these two standard forms can somehow be combined. The Kalman decomposition does exactly that.

Suppose (A, B, C, D) are the matrices that specify the given n^{th}-order LTI state-space model, and suppose we construct a transformation matrix

$$T = \begin{bmatrix} T_{r\overline{o}} & T_{ro} & T_{\overline{r}\overline{o}} & T_{\overline{r}o} \end{bmatrix} \tag{25.1}$$

where the submatrices are defined as follows:

1. The columns of $T_{r\overline{o}}$ form a basis for $\mathbb{R} \cap \overline{\mathbb{O}}$, the subspace that is both reachable and unobservable (verify that the intersection of two subspaces is a subspace);
2. T_{ro} complements $T_{r\overline{o}}$ in the reachable subspace, so that $Ra \begin{bmatrix} T_{r\overline{o}} & T_{ro} \end{bmatrix} = \mathbb{R}$;
3. $T_{\overline{r}\overline{o}}$ complements $T_{r\overline{o}}$ in the unobservable subspace, so that $Ra \begin{bmatrix} T_{r\overline{o}} & T_{\overline{r}\overline{o}} \end{bmatrix} = \overline{\mathbb{O}}$;
4. $T_{\overline{r}o}$ complements $\begin{bmatrix} T_{r\overline{o}} & T_{ro} & T_{\overline{r}\overline{o}} \end{bmatrix}$ to span $\mathbb{R}^n$, so that T is invertible.

Of course, any of these matrices may turn out to be of dimension 0, *e.g.* when the system is both reachable and observable, the matrix T_{ro} is $n \times n$, and all the other submatrices disappear. We now perform a similarity transformation using T, thereby carrying out the mapping

$$(A, B, C, D) \longrightarrow \left(T^{-1}AT, T^{-1}B, CT, D\right) = (\widehat{A}, \widehat{B}, \widehat{C}, D).$$

The system $(\widehat{A}, \widehat{B}, \widehat{C}, D)$ is said to be in Kalman decomposed form. This is a standard form that has a very illuminating structure, which we will now deduce based on the form of the T matrix and the following additional constraints:

$$\begin{aligned} A\mathbb{R} &\subseteq \mathbb{R} && (25.2)\\ A\overline{\mathbb{O}} &\subseteq \overline{\mathbb{O}} && (25.3)\\ Ra(B) &\subseteq \mathbb{R} && (25.4)\\ \overline{\mathbb{O}} &\subseteq \text{Null}(C). && (25.5) \end{aligned}$$

Equations (25.2) and (25.3) simply restate the fact that the reachable and unobservable subspaces are A-invariant. To determine the form of $\widehat{A}$, we begin by writing

$$AT = T\widehat{A}$$

which can be expanded into

$$A\left[\begin{array}{cccc} T_{r\overline{o}} & T_{ro} & T_{\overline{r}\overline{o}} & T_{\overline{r}o} \end{array}\right] = \left[\begin{array}{cccc} T_{r\overline{o}} & T_{ro} & T_{\overline{r}\overline{o}} & T_{\overline{r}o} \end{array}\right] \left[\begin{array}{cccc} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{array}\right]. \qquad (25.6)$$

From (25.2) and (25.3), we have that the range of $AT_{r\overline{o}}$ remains in $Ra(T_{r\overline{o}})$, the space that is both reachable and unobservable. From (25.6),

$$AT_{r\overline{o}} = \left[\begin{array}{cccc} T_{r\overline{o}} & T_{ro} & T_{\overline{r}\overline{o}} & T_{\overline{r}o} \end{array}\right] \left[\begin{array}{c} A_{11} \\ A_{21} \\ A_{31} \\ A_{41} \end{array}\right],$$

so we must have A_{21}, A_{31} and $A_{41} = 0$. Similarly, from (25.2) we deduce that A_{32} and A_{42} must be zero. From (25.3), it follows that A_{23} and A_{43} are zero. By applying all of these conditions (and with a notational change in the subscripts), we arrive at the final form of $\widehat{A}$:

$$\widehat{A} = \left[\begin{array}{cccc} A_{r\overline{o}} & A_{12} & A_{13} & A_{14} \\ 0 & A_{ro} & 0 & A_{24} \\ 0 & 0 & A_{\overline{r}\overline{o}} & A_{34} \\ 0 & 0 & 0 & A_{\overline{r}o} \end{array}\right]. \qquad (25.7)$$

Proceeding with the same line of logic, and noting conditions (25.4) and (25.5), we have

$$B = T\widehat{B} = \left[\begin{array}{cccc} T_{r\overline{o}} & T_{ro} & T_{\overline{r}\overline{o}} & T_{\overline{r}o} \end{array}\right] \left[\begin{array}{c} B_{r\overline{o}} \\ B_{ro} \\ 0 \\ 0 \end{array}\right], \qquad (25.8)$$

and, from $CT = \widehat{C}$,

$$C\left[\begin{array}{cccc} T_{r\overline{o}} & T_{ro} & T_{\overline{r}\overline{o}} & T_{\overline{r}o} \end{array}\right] = \left[\begin{array}{cccc} 0 & C_{ro} & 0 & C_{\overline{r}o} \end{array}\right]. \qquad (25.9)$$

In the resulting Kalman-decomposed form $(\widehat{A}, \widehat{B}, \widehat{C}, D)$, the subsystem $(A_{ro}, B_{ro}, C_{ro}, D)$ is both reachable and observable (prove this!). Similarly, the reachable subsystem is

$$\left(\begin{bmatrix} A_{r\overline{o}} & A_{12} \\ 0 & A_{ro} \end{bmatrix}, \begin{bmatrix} B_{r\overline{o}} \\ B_{ro} \end{bmatrix}, \begin{bmatrix} 0 & C_{ro} \end{bmatrix}, D \right)$$

with its unobservable portion already displayed in standard form, and the observable subsystem is

$$\left(\begin{bmatrix} A_{ro} & A_{24} \\ 0 & A_{\overline{r}o} \end{bmatrix}, \begin{bmatrix} B_{ro} \\ 0 \end{bmatrix}, \begin{bmatrix} C_{ro} & C_{\overline{r}o} \end{bmatrix}, D \right)$$

with its reachable portion already displayed in standard form. The Figure 25.1 constitutes a representation of the system $(\widehat{A}, \widehat{B}, \widehat{C}, D)$:

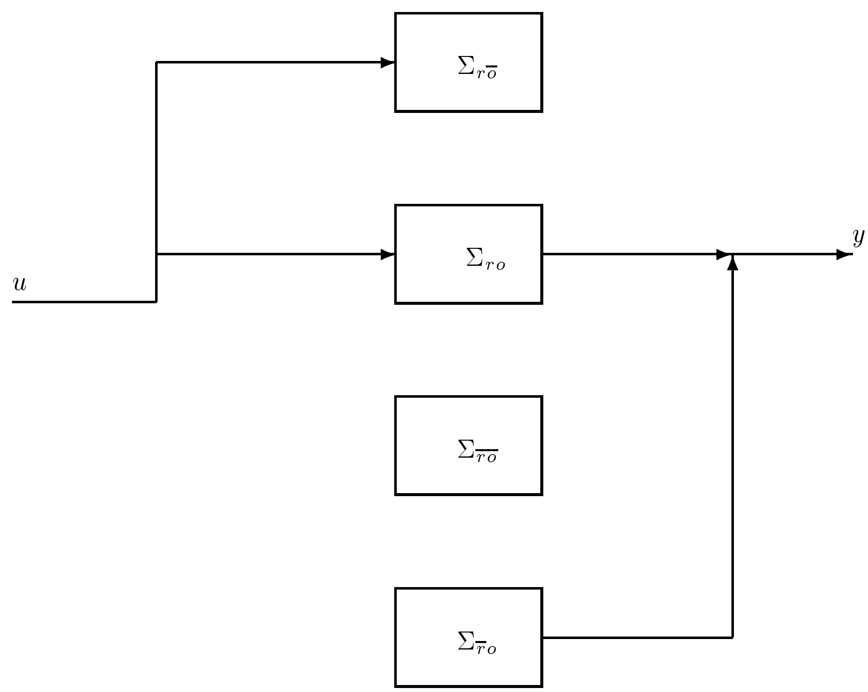


Figure 25.1: Kalman Decomposition of a State Space Model.

As can be shown quite easily, the Kalman decomposition is unique up to a similarity transformation that has the same block structure as $\widehat{A}$. (To show this for yourself, first prove that the columns of full-column-rank matrices P, Q are bases for the same space iff $P = QM$ for some invertible matrix M.) It follows that:

- the matrices $A_{r\overline{o}}$, A_{ro}, $A_{\overline{ro}}$, $A_{\overline{r}o}$ are uniquely defined up to a similarity transformation — their eigenvalues (and indeed their Jordan structure) are thus uniquely defined, and may be classified as $r\overline{o}$, ro, $\overline{ro}$, $\overline{r}o$ respectively;
- the ro subsystem (as also the reachable subsystem and the observable subsystem) is uniquely defined up to similarity.

It is clear from the Kalman decomposition and the associated figure above that the input/output behavior of the system for zero initial conditions is determined entirely by the ro part of the system. Also, the output behavior for arbitrary input and initial conditions is determined by the observable part of the system.

25.3 State-Space Realizations of Transfer Functions

Given a DT LTI state-space model (A, B, C, D), we have seen that its transfer function is simply

$$H(z) = C(zI - A)^{-1}B + D.$$

(For a CT system (A, B, C, D), we obtain the same expression for the transfer function, except that z is replaced by s.) For a MIMO system with m inputs and p outputs, this results in a $p \times m$ matrix of rational functions of z (or s, in CT). Recall that $H(z)$ is in general proper (*i.e.,* all entries have numerator degree less than or equal to the degree of the denominator), and for $|z| \to \infty$, we have $H(z) \to D$ (so the transfer function is *strictly* proper if $D = 0$).

Now consider the converse problem. Given a transfer function, can one always find a state-space representation? This is called the *realization* problem.

Definition 25.1 *(A, B, C, D) is called a* **realization** *of the transfer function $H(z)$ if*

$$H(z) = C(zI - A)^{-1}B + D.$$

To phrase the above problem in the time domain, expand $H(z)$ as

$$H(z) = H_0 + z^{-1}H_1 + z^{-2}H_2 + \dots \tag{25.10}$$

In the SISO DT case, we know that $H_0,\ H_1,\ H_2,\ \dots$ constitute the output response at time $0,\ 1,\ 2, \dots$ to a unit sample at time 0 applied to the input of the system when it is at rest ($x(0) = 0$), *i.e.* the sequence $\{H_k\}$ is the unit-sample response or "impulse" response of the system. In the MIMO case, the interpretation is similar, except that now the ij^{th} entry of H_k is the value at time k of the zero-state response at the i^{th} output to a unit impulse at the j^{th} input. (The H_k are also referred to as *Markov parameters.*) For the state-space model $(A,\ B,\ C,\ D)$, it is straightforward to see that

$$\begin{aligned} H_0 &= D, \\ H_k &= CA^{k-1}B, \qquad k \geq 1 \end{aligned} \tag{25.11}$$

This can be verified directly in the time domain, or by expanding $(zI - A)^{-1}$ in (25.3) as

$$(zI - A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \cdots \tag{25.12}$$

(an expansion that is valid for $|z|$ greater than the spectral radius of A) and then equating the coefficients of z^{-k} with those in the expression (25.10). The realization problem, i.e. the problem of finding $(A,\ B,\ C,\ D)$ such that (25.3) holds, can now be rephrased equivalently as that of finding a state-space model $(A,\ B,\ C,\ D)$ such that the relations in (25.11) hold.

It is evident that state-space realizations are not unique. For instance, given one realization, we can obtain an infinite number of realizations through similarity transformations. (You should verify algebraically that this is indeed the case.) However, the Kalman decomposition makes clear that there are still other possible realizations. Specifically, you should verify that

$$\begin{aligned} H(z) &= \widehat{C}(zI - \widehat{A})^{-1}\widehat{B} + D \\ &= C_{ro}(zI - A_{ro})^{-1}B_{ro} + D \end{aligned} \tag{25.13}$$

i.e. only the ro part of a system contributes to its transfer function, so if a given realization is not ro, then its ro subsystem (or any similarity transformation of it) constitutes an alternative realization of $H(z)$. Going in the other direction, one can obtain a new realization from a given one by adding unreachable and/or unobservable dynamics. Thus, different realizations of $H(z)$ can differ in their orders. A *minimal* realization is one of *least possible order.*

25.4 Minimal Realizations

SISO Systems

To get some feel for how realizations relate to transfer functions, consider a SISO system in controller canonical form:

$$\widetilde{A} = \begin{bmatrix} -a_1 & \cdots & \cdots & -a_n \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix}, \quad \widetilde{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{25.14}$$

$$\widetilde{c} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}, \qquad d.$$

(You should draw yourself a block diagram of this, using delays, adders, gains.) Now verify that its transfer function is

$$H(z) = \frac{c_1 z^{n-1} + \cdots + c_n}{z^n + a_1 z^{n-1} + \cdots + a_n} + d \tag{25.15}$$

We can argue quite easily that there is a realization of order $< n$ for this $H(z)$ iff the numerator and denominator polynomials, $c(z) = c_1 z^{n-1} + \cdots + c_n$ and $a(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ respectively, have a common factor that cancels out. (If there is such a factor, we can get a controller canonical form realization of order $< n$, by inspection. Conversely, if there is a realization of order $< n$, then its transfer function will have denominator degree $< n$, which implies that $c(z)$, $a(z)$ above have a common factor.)

Now, a common factor $(z - \lambda)$ between $c(z)$ and $a(z)$ exists iff

$$\begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix} = 0 \tag{25.16}$$

for some λ that is a root of $a(z) = |zI - \widetilde{A}|$, *i.e.* for some λ that is an eigenvalue of $\widetilde{A}$. Verifying that the column vector in the preceding equation is the corresponding eigenvector of $\widetilde{A}$, we recognize from the modal test for observability that the condition in this equation is precisely equivalent to *unobservability* of the controller-form realization. We are now in a position to prove the following result:

Theorem 25.1 *A state-space realization of a SISO transfer function $H(z)$ is* **minimal** *iff it is reachable and observable.*

Proof.
If the realization is not ro, then the ro part of its Kalman decomposition will yield a lower-order realization, which means the original realization was not minimal.

Conversely, if the realization is reachable and observable, it can be transformed to controller canonical form, and the denominator $|zI - \widetilde{A}|$ of $H(z)$ will have no cancellations with the numerator, so the realization will be minimal.

MIMO Systems

The preceding theorem also holds for the MIMO case, as we shall demonstrate now. Our proof of the MIMO result will use a different route than what was used in the SISO case, because a proof analogous to the SISO one would rely on machinery — such as matrix fraction descriptions of rational matrices — which we shall not be developing for the MIMO case in this course. There is nevertheless some value in seeing the SISO arguments above, because they provide additional insight into what is going on.

Theorem 25.2 *A realization is minimal iff it is reachable and observable.*

Proof. If a realization is not reachable or not observable, we can use the Kalman decomposition to extract its ro part, and thereby obtain a realization of smaller order.

For the converse, suppose $(A,\ B,\ C,\ D)$ is a reachable, observable realization of order n, but is not minimal. Then there is a minimal realization $(A^*,\ B^*,\ C^*,\ D^*)$ of order $n^* < n$ (and necessarily reachable and observable, from the first part of our proof). Now

$$\begin{aligned}\mathcal{O}_n R_n &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1}\end{bmatrix}\begin{bmatrix} B & AB & \dots A^{n-1}B\end{bmatrix} \\ &= \begin{bmatrix} H_1 & H_2 & \cdots & H_n \\ H_2 & & \diagup & \vdots \\ \vdots & & & \vdots \\ H_n & \cdots & \cdots & H_{2n-1}\end{bmatrix} = \mathcal{O}_n^* R_n^* \qquad (25.17)\end{aligned}$$

The reachability and observability of $(A,\ B,\ C,\ D)$ ensures that $\text{rank}(\mathcal{O}_n R_n) = n$ (as can be verified using Sylvester's inequality) while $\text{rank}(\mathcal{O}_n^* R_n^*) = \text{rank}(\mathcal{O}_{n^*}^* R_{n^*}^*) = n^*$, but then (25.17) is impossible. Hence there is no realization of order less than n if there is a reachable and observable one of order n.

The following theorem shows that minimal realizations are tightly connected; in fact there is in effect only *one* minimal realization of a given $H(z)$, up to a similarity transformation (or change of coordinates)!

Theorem 25.3 *All minimal realizations of a given transfer function are similar to each other.*

Proof. Suppose (A, B, C, D) and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are two minimal realizations of order n. Then $D = \tilde{D}$ and $CA^kB = \tilde{C}\tilde{A}^k\tilde{B}, \;\; k \geq 0$, so

$$\mathcal{O}_n R_n = \tilde{\mathcal{O}}_n \tilde{R}_n. \tag{25.18}$$

Also

$$\mathcal{O}_n A R_n = \tilde{\mathcal{O}}_n \tilde{A} \tilde{R}_n \tag{25.19}$$

Let us introduce the notation M^+ to denote the ("Moore-Penrose") *pseudo-inverse* of a matrix M. If M has full column rank, then $M^+ = (M'M)^{-1}M'$, while if M has full row rank, then $M^+ = M'(MM')^{-1}$ (and in the general case the pseudo-inverse can be explicitly written in terms of the SVD of M, but we shall not need this case for the proof). It is then easy to verify from (25.18) that

$$R_n \tilde{R}_n^+ = O_n^+ \tilde{\mathcal{O}}_n = T \tag{25.20}$$

and that

$$T^{-1} = \tilde{\mathcal{O}}_n^+ \mathcal{O}_n = \tilde{R}_n R_n^+ \tag{25.21}$$

(You should note how the reachability and observability of the minimal realizations are invoked to make the necessary arguments.) It is then easy to check, using (25.18) and (25.19) that

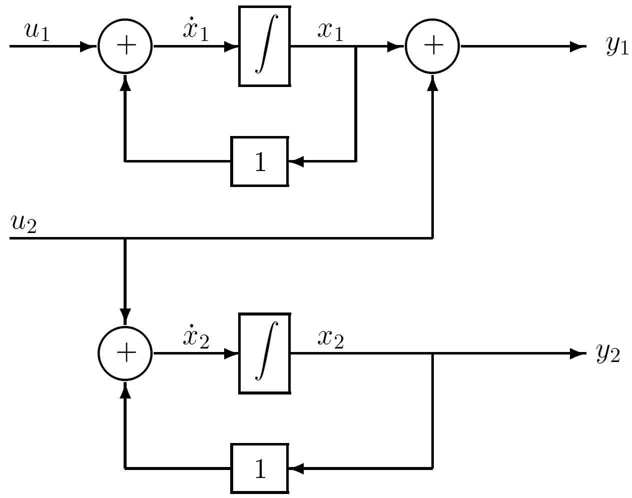
$$AT = T\tilde{A}, \; B = T\tilde{B}, \; \tilde{C} = CT \tag{25.22}$$

i.e. the realizations are similar.

All of the above results carry over to the CT case. The only modification is in the interpretation of the Markov parameters; a CT interpretation can be found in terms of moments of the impulse response, but is not particularly interesting.

We have seen how to obtain realizations of SISO transfer functions, by building on canonical forms. The situation is more involved for MIMO transfer functions. One brute-force realization approach would be to simply realize all of the SISO elements $h_{ij}(s)$ of $H(s)$, and then connect them to form the outputs.

Example 25.1 (We use a CT system in this example to make the point that all the preceding development carries over unchanged to the CT case.) The 2×2 transfer function $H(s) = \begin{bmatrix} \frac{1}{s-1} & 1 \\ 0 & \frac{1}{s-1} \end{bmatrix}$ can be immediately realized in state-space form by constructing (minimal) realizations of the individual entries of $H(s)$ and interconnecting them as needed:



The corresponding state-space model is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and this is easily verified to be reachable and observable, hence minimal. However, the component-wise realization procedure is *not* guaranteed to produce a minimal realization. For instance, with

$$H(s) = \left[\begin{array}{cc} \frac{1}{s-1} & \frac{1}{s-1} \end{array} \right],$$

combining component-wise realizations into an overall realization would lead to a second-order realization, but there is a (minimal) realization of order 1 (which you should find!).

Exercise 25.4 guides you through a general procedure for the construction of a minimal realization *if the minimal order is known*, using the Markov parameters computed from the transfer function. Following, we describe another approach ("Gilbert's method") that is based on the residues at the poles of the transfer matrix.

Gilbert's Realization

Suppose we have a proper matrix transfer function $H(z)$, and we factor out the polynomial $d(z)$ that is the least common denominator of all the entries of $H(z)$ (i.e. the least common multiple of the denominators of all the entries). If $d(z)$ has no repeated roots, then it is possible to construct a minimal realization via Gilbert's method. (There is a generalization for repeated poles, but we omit it.) First apply a partial fraction expansion to each of the elements of $H(z)$ and collect residues for each distinct pole. Denoting the q roots of $d(z)$ by $\rho_1, \cdots, \rho_q$, we can write the transfer function matrix in the following form:

$$H(z) = D + \sum_{i=1}^{q} \frac{1}{z - \rho_i} R_i$$

where R_i is also $p \times m$ and $D = H(\infty)$. Let us denote the rank of R_i by r_i; it will turn out that r_i is the minimum number of poles with location ρ_i required to realize $H(z)$. Since the rank of R_i is r_i, this matrix can be decomposed as the product of two matrices with full column and row rank, respectively, each with rank r_i:

$$R_i = C_i^{p \times r_i} B_i^{r_i \times m}, \ \ \mathrm{rank}(R_i) = r_i$$

It is now easy to verify that $H(z) = C(zI - A)^{-1}B + D$, where

$$A = \begin{bmatrix} \underbrace{\begin{matrix} \rho_1 & & \\ & \ddots & \\ & & \rho_1 \end{matrix}}_{r_1} & & 0 \\ & \ddots & \\ 0 & & \underbrace{\begin{matrix} \rho_q & & \\ & \ddots & \\ & & \rho_q \end{matrix}}_{r_q} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_q \end{bmatrix}$$

$$C = \begin{bmatrix} C_1 & C_2 & \cdots & C_q \end{bmatrix}, \qquad D\,.$$

This realization is easily verified to be reachable and observable, hence minimal.

Exercises

Exercise 25.1 Find a state-space description of the circuit below, in the form $\dot{x}(t) = Ax(t) + Bi(t)$, with output equation $v(t) = Cx(t) + Di(t)$, choosing i_L and v_C as state variables, and with R_1, R_2, L and C all equal to 1.

(a) Is the system controllable? Is it observable? What is its transfer function? (Evaluate the transfer function using the state-space description, and make sure that all common factors between numerator and denominator are cancelled. Then check your answer by direct impedance calculations with the circuit.)

(b) What are the eigenvalues and the left and right eigenvectors of A? Is A diagonalizable? Also verify that your eigenvectors are consistent with your conclusions regarding controllability and observability in (a).

(c) By carefully interpreting the results of (a) and (b), or by explicitly computing the Kalman decomposition, determine how many eigenvalues of A are in each of the following categories:

(i) co: controllable and observable;

(ii) $c\bar{o}$: controllable and unobservable;

(iii) $\bar{c}\bar{o}$: uncontrollable and unobservable;

(iv) $\bar{c}o$: uncontrollable and observable.

(d) Only *one* of the following equations (for some appropriate choice of the parameters) *precisely* represents the set of voltage waveforms $v(t)$ that are possible for this circuit, assuming *arbitrary* initial conditions. Determine which one, and specify the coefficients, stating your reasoning.

(i) $v(t) = \alpha i(t)$;

(ii) $[dv(t)/dt] + \beta v(t) = [di(t)/dt] + \alpha i(t)$;

(iii) $[d^2v(t)/dt^2] + \delta[dv(t)/dt] + \beta v(t) = [d^2i(t)/dt^2] + \gamma[di(t)/dt] + \alpha i(t)$.

Exercise 25.2 **(a)** Find a third-order state-space realization in controller canonical form for the transfer function $H_1(s) = (s + f)/(s + 4)^3$, where f is a parameter. (To do this, assume the "A" and "b" of the state-space model are in controller form, then find what "c" and "d" need to be to make the transfer function come out right.) For what values of f does your model lose (i) observability? (ii) controllability?

Similarly, find a first-order controller canonical form realization of the transfer function $H_2(s) = 1/(s-2)$.

(b) Now suppose the realizations in (a) are connected in cascade, with the output of the first system used as the input to the second. The input to the first system then becomes the overall system input, and the output of the second system becomes the overall system output:

$$u \longrightarrow H_1(s) \longrightarrow H_2(s) \longrightarrow y$$

Write down a fourth-order state-space description of the cascade. Is the cascaded system asymptotically stable? — and does your answer depend on f?

Now determine for what values of f the cascaded system loses (i) observability, (ii) controllability. Interpret your results in terms of pole-zero cancellations between $H_1(s)$ and $H_2(s)$. Is there a value of f for which the cascaded system is bounded-input/bounded-output (BIBO) stable but *not* asymptotically stable.

Exercise 25.3 Suppose a least one eigenvalue of the $n \times n$ matrix A is at 0, and that this eigenvalue is reachable with input vector b and observable with output vector c. Show that $A + bgc$, for *any nonzero* g, has *no* eigenvalues at 0.

Exercise 25.4 You are given the Markov parameters $\{H_i\}$ associated with a particular $p \times m$ transfer matrix $H(z) = H_0 + z^{-1}H_1 + z^{-2}H_2 + \cdots$, and you are told that all minimal realizations of $H(z)$ are of a given order n. This problem aims at finding a minimal realization from the Markov parameters.

Let $x(k+1) = Ax(k) + Bu(k)\ ,\ y(k) = Cx(k) + Du(k)$ denote some specific, but unkown, minimal realization of $H(z)$, with $\mathcal{B}_n$ and $\mathcal{C}_n$ denoting its reachability and observability matrices respectively. (For notational convenience, we shall drop the subscript n in what follows.) We shall construct a realization of $H(z)$ that will be shown to be *similar* to this minimal realization, and therefore itself minimal. The following two matrices (with "block-Hankel" structure) will be needed for this problem:

$$K_1 = \begin{pmatrix} H_1 & H_2 & \cdots & H_n \\ H_2 & H_3 & \cdots & H_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ H_n & H_{n+1} & \cdots & H_{2n-1} \end{pmatrix}$$

$$K_2 = \begin{pmatrix} H_2 & H_3 & \cdots & H_{n+1} \\ H_3 & H_4 & \cdots & H_{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ H_{n+1} & H_{n+2} & \cdots & H_{2n} \end{pmatrix}$$

(a) Show that $K_1 = \mathcal{C}\mathcal{B}$ and $K_2 = \mathcal{C}A\mathcal{B}$.

[**(b)**] Show that K_1 has rank n.

[**(c)**] We can decompose K_1 (for example using its SVD) into a product LR, where the left factor L has full column rank ($= n$, from (b)), and the right factor has full row rank ($= n$ also). Show that $\mathcal{B} = T_1 R$ and $\mathcal{C} = LT_2$ for some *nonsingular* matrices T_1 and T_2, and prove that $T_2 = T_1^{-1}$.

(d) Define C_1 to be the matrix formed from the first p rows of L, and show that $C_1 = CT_1$. Similarly, define B_1 to be the matrix formed from the first m columns of R, and show that $B_1 = T_1^{-1}B$.

(e) Define $A_1 = L^+ K_2 R^+$, where the superscript $+$ denotes the pseudo-inverse of the associated matrix, and show that $A_1 = T_1^{-1} A T_1$.

The desired minimal realization is now (A_1, B_1, C_1, D_1), where $D_1 = H_0$.

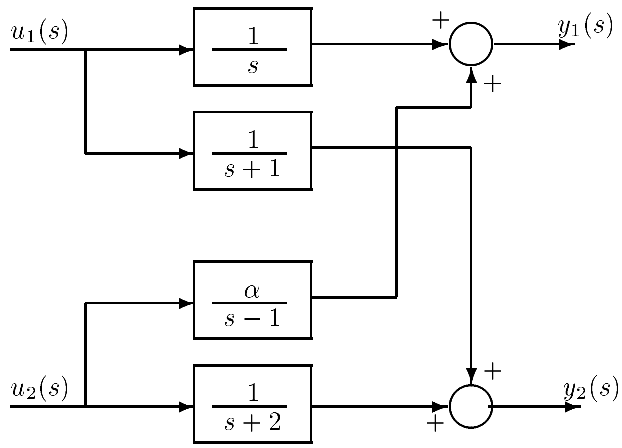
Exercise 25.5 **(a)** Obtain a minimal realization of the system:

$$H(s) = \begin{bmatrix} \frac{s}{(s-1)^2} & \frac{1}{(s-1)} \\ \\ \frac{-6}{(s-1)(s+3)} & \frac{1}{(s+3)} \end{bmatrix}.$$

Explicitly verify its minimality.

(b) Compute the poles (including multiplicities) of this transfer function using the minimal realization you obtained.

Exercise 25.6 The two-input, two-output system below is obtained by interconnecting four SISO subsystems as shown. (Note, incidentally, that none of the SISO transfer functions has any zeros.) The scalar gain α is real and nonzero, but can be either positive or negative.



(a) Assemble *minimal* state-space realizations of the SISO subsystems into an overall state-space description of the two-input, two-output system. Determine whether the resulting system is reachable and observable, and also find its natural frequencies.

(b) Determine the transfer function matrix $G(s)$ that relates the two outputs to the two inputs. How do the poles of $G(s)$ relate to the natural frequencies that you found in (a)?

(c) Compute the number and locations of the MIMO transmission zeros as a function of α, by finding expressions for the frequencies at which $G(s)$ loses rank. Are there any allowed (i.e. nonzero) values of α that yield transmission zeros at the same locations as poles?

(d) Now set $\alpha = +1$. Determine the transmission zero location $s = \zeta$ and the corresponding input direction u_0 from the null space of the matrix $G(\zeta)$. Now obtain the analytical solution to the state equations for arbitrary values of the initial state at time 0, as well as the corresponding analytical expressions for the two outputs $y_1(t)$ and $y_2(t)$, when the system is driven by the specific input $u(t) = u_0 e^{\zeta t}$ for $t \geq 0$. (Note that the expressions for the outputs do not contain the zero-frequency term $e^{\zeta t}$; it has been "absorbed" by the system.) Also determine what initial state would yield both $y_1(t) = y_2(t) = 0$ for all $t \geq 0$, with this particular input.

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