Lectures on Dynamic Systems and Control

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Chapter 10

Discrete-Time Linear State-Space Models

10.1 Introduction

In the previous chapters we showed how dynamic models arise, and studied some special characteristics that they may possess. We focused on state-space models and their properties, presenting several examples. In this chapter we will continue the study of state-space models, concentrating on solutions and properties of DT *linear* state-space models, both time-varying and time-invariant.

10.2 Time-Varying Linear Models

A general nth-order discrete-time linear state-space description takes the following form:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ y(k) &= C(k)x(k) + D(k)u(k) \ , \end{aligned} \tag{10.1}$$

where $x(k) \in \mathbb{R}^n$. Given the initial condition $x(0)$ and the input sequence $u(k)$, we would like to find the state sequence or state *trajectory* $x(k)$ as well as the output sequence $y(k)$.

Undriven Response

First let us consider the *undriven response*, that is the response when $u(k) = 0$ for all $k \in \mathbb{Z}$. The state evolution equation then reduces to

$$x(k+1) = A(k)x(k) \ . \tag{10.2}$$

The response can be derived directly from (10.2) by simply iterating forward:

$$\begin{aligned} x(1) &= A(0)x(0) \\ x(2) &= A(1)x(1) \\ &= A(1)A(0)x(0) \\ x(k) &= A(k-1)A(k-2)\ldots A(1)A(0)x(0) \end{aligned} \tag{10.3}$$

Motivated by (10.3), we define the **state transition matrix**, which relates the state of the undriven system at time k to the state at an earlier time ℓ:

$$x(k) = \Phi(k,\ell)x(\ell) \quad k \geq \ell \,. \tag{10.4}$$

The form of the matrix follows directly from (10.3):

$$\Phi(k,\ell) = \begin{cases} A(k-1)A(k-2)\cdots A(\ell) & , \quad k > \ell \geq 0 \\ I & , \quad k = \ell \end{cases} \,. \tag{10.5}$$

If $A(k-1)$, $A(k-2)$,..., $A(\ell)$ are all invertible, then one could use the state transition matrix to obtain $x(k)$ from $x(\ell)$ even when $k < \ell$, but we shall typically assume $k \geq \ell$ when writing $\Phi(k,\ell)$.

The following properties of the discrete-time state transition matrix are worth highlighting:

$$\begin{aligned} \Phi(k,k) &= I \\ x(k) &= \Phi(k,0)x(0) \\ \Phi(k+1,\ell) &= A(k)\Phi(k,\ell). \end{aligned} \tag{10.6}$$

Example 10.1 (A Sufficient Condition for Asymptotic Stability)

The linear system (10.1) is termed *asymptotically stable* if, with $u(k) \equiv 0$, and for all $x(0)$, we have $x(n) \to 0$ (by which we mean $\|x(n)\| \to 0$) as $n \to \infty$. Since $u(k) \equiv 0$, we are in effect dealing with (10.2).

Suppose

$$\|A(k)\| \leq \gamma < 1 \tag{10.7}$$

for all k, where the norm is any submultiplicative norm and γ is a constant (independent of k) that is less than 1. Then

$$\|\Phi(n,0)\| \leq \gamma^n$$

and hence

$$\|x(n)\| \leq \gamma^n \|x(0)\|$$

so $x(n) \to 0$ as $n \to \infty$, no matter what $x(0)$ is. Hence (10.7) constitutes a sufficient condition (though a weak one, as we'll see) for asymptotic stability of (10.1).

Example 10.2 ("Lifting" a Periodic Model to an LTI Model)

Consider an undriven *linear, periodically varying* (LPV) model in state-space form. This is a system of the form (10.2) for which there is a smallest positive integer N such that $A(k+N) = A(k)$ for all k; thus N is the *period* of the system. (If $N = 1$, the system is actually LTI, so the cases of interest here are really those with $N \geq 2$.) Now focus on the state vector $x(mN)$ for integer m, i.e., the state of the LPV system sampled regularly once every period. Evidently

$$\begin{aligned} x(mN+N) &= \Big[A(N-1)A(N-2)\cdots A(0) \Big]\, x(mN) \\ &= \Phi(N,0)\, x(mN) \end{aligned} \tag{10.8}$$

The sampled state thus admits an LTI state-space model. The process of constructing this sampled model for an LPV system is referred to as *lifting*.

Driven Response

Now let us consider the driven system, *i.e.*, $u(k) \neq 0$ for at least some k. Referring back to (10.1), we have

$$\begin{aligned} x(1) &= A(0)x(0) + B(0)u(0) \\ x(2) &= A(1)x(1) + B(1)u(1) \\ &= A(1)A(0)x(0) + A(1)B(0)u(0) + B(1)u(1) \end{aligned} \tag{10.9}$$

which leads to

$$\begin{aligned} x(k) &= \Phi(k,0)x(0) + \sum_{\ell=0}^{k-1} \Phi(k,\ell+1)B(\ell)u(\ell) \\ &= \Phi(k,0)x(0) + \Gamma(k,0)\mathcal{U}(k,0)\ , \end{aligned} \tag{10.10}$$

where

$$\Gamma(k,0) = \Big[\Phi(k,1)B(0) \mid \Phi(k,2)B(1) \mid \cdots \mid B(k-1) \Big]\ ,\ \ \mathcal{U}(k,0) = \begin{pmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{pmatrix} \tag{10.11}$$

What (10.10) shows is that the solution of the system over k steps has the same form as the solution over one step, which is given in the first equation of (10.1). Also note that the system response is divided into two terms: one depends only on the initial state $x(0)$ and the other depends only on the input. These terms are respectively called the *natural* or *unforced* or *zero-input* response, and the *zero-state* response. Note also that the zero-state response has a form that is reminiscent of a convolution sum; this form is sometimes referred to as a *superposition sum*.

If (10.10) had been simply claimed as a solution, without any sort of derivation, then its validity could be verified by substituting it back into the system equations:

$$\begin{aligned}
x(k+1) &= \Phi(k+1,0)x(0) + \sum_{\ell=0}^{k} \Phi(k+1,\ell+1)B(\ell)u(\ell) \\
&= \Phi(k+1,0)x(0) + \sum_{\ell=0}^{k-1} \Phi(k+1,\ell+1)B(\ell)u(\ell) + B(k)u(k) \\
&= A(k)\left[\Phi(k,0)x(0) + \sum_{\ell=0}^{k-1} \Phi(k,\ell+1)B(\ell)u(\ell)\right] + B(k)u(k) \\
&= A(k)x(k) + B(k)u(k)\ .
\end{aligned} \tag{10.12}$$

Clearly, (10.12) satisfies the system equations (10.1). It remains to be verified that the proposed solution matches the initial state at $k = 0$. We have

$$x(0) = \Phi(0,0)x(0) = x(0)\ , \tag{10.13}$$

which completes the check.

If $\mathcal{Y}(k,0)$ is defined similarly to $\mathcal{U}(k,0)$, then following the sort of derivation that led to (10.10), we can establish that

$$\mathcal{Y}(k,0) = \Theta(k,0)x(0) + \Psi(k,0)\mathcal{U}(k,0) \tag{10.14}$$

for appropriately defined matrices $\Theta(k,0)$ and $\Psi(k,0)$. We leave you to work out the details. Once again, (10.14) for the output over k steps has the same form as the expression for the output at a single step, which is given in the second equation of (10.1).

10.3 Linear Time-Invariant Models

In the case of a *time-invariant* linear discrete-time system, the solutions can be simplified considerably. We first examine a direct time-domain solution, then compare this with a transform-domain solution, and finally return to the time domain, but in modal coordinates.

Direct Time-Domain Solution

For a linear time-invariant system, observe that

$$\left.\begin{aligned} A(k) &= A \\ B(k) &= B \end{aligned}\right\} \quad \text{for all} \quad k \geq 0, \tag{10.15}$$

where A and B are now constant matrices. Thus

$$\Phi(k,\ell) = A(k-1)\ldots A(\ell) = A^{k-\ell}\ , \quad k \geq \ell \tag{10.16}$$

so that, substituting this back into (10.10), we are left with

$$\begin{aligned} x(k) &= A^k x(0) + \sum_{\ell=0}^{k-1} A^{k-\ell-1} B u(\ell) \\ &= A^k x(0) + \left[A^{k-1}B \mid A^{k-2}B \mid \cdots \mid B \right] \begin{pmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{pmatrix} \end{aligned} \tag{10.17}$$

Note that the zero-state response in this case exactly corresponds to a convolution sum. Similar expressions can be worked out for the outputs, by simplifying (10.14); we leave the details to you.

Transform-Domain Solution

We know from earlier experience with dynamic linear time-invariant systems that the use of appropriate transform methods can reduce the solution of such a system to the solution of algebraic equations. This expectation does indeed hold up here. First recall the definition of the one-sided $\mathcal{Z}$-transform:

Definition 10.1 *The one-sided $\mathcal{Z}$-transform, $F(z)$, of the sequence $f(k)$ is given by*

$$F(z) = \sum_{k=0}^{\infty} z^{-k} f(k)$$

for all z such that the result of the summation is well defined, denoted by the Region of Convergence (ROC).

The sequence $f(k)$ can be a vector or matrix sequence, in which case $F(z)$ is respectively a vector or matrix as well.

It is easy to show that the transform of a sum of two sequences is the sum of the individual transforms. Also, scaling a sequence by a constant simply scales the transform by the same constant. The following shift property of the one-sided transform is critical, and not hard to establish. Suppose that $f(k) \overset{\mathcal{Z}}{\longleftrightarrow} F(z)$. Then

1.

$$g(k) = \begin{cases} f(k-1) & ; \quad k \geq 1 \\ 0 & ; \quad k = 0 \end{cases} \quad \Longrightarrow \quad G(z) = z^{-1} F(z).$$

2.

$$g(k) = f(k+1) \quad \Longrightarrow \quad G(z) = z\left[F(z) - f(0)\right].$$

Convolution is an important operation that can be defined on two sequences $f(k)$, $g(k)$ as

$$f * g(k) = \sum_{m=0}^{k} g(k-m)f(m),$$

whenever the dimensions of f and g are compatible so that the products are defined. The $\mathcal{Z}$ transform of a convolutions of two sequences satisfy

$$\begin{aligned}
\mathcal{Z}(f * g) &= \sum_{k=0}^{\infty} z^{-k} f * g(k) \\
&= \sum_{k=0}^{\infty} z^{-k} \left(\sum_{m=0}^{k} f(k-m)g(m) \right) \\
&= \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} z^{-k} f(k-m)g(m) \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} z^{-(k+m)} f(k)g(m) \\
&= \sum_{m=0}^{\infty} z^{-m} \left(\sum_{k=0}^{\infty} z^{-k} f(k) \right) g(m) \\
&= F(z)G(z).
\end{aligned}$$

Now, given the state-space model (10.1), we can take transforms on both sides of the equations there. Using the transform properties just described, we get

$$zX(z) - zx(0) = AX(z) + BU(z) \tag{10.18}$$

$$Y(z) = CX(z) + DU(z). \tag{10.19}$$

This is solved to yield

$$\begin{aligned}
X(z) &= z(zI-A)^{-1}x(0) + (zI-A)^{-1}BU(z) \\
Y(z) &= zC(zI-A)^{-1}x(0) + \underbrace{\left[C(zI-A)^{-1}B + D \right]}_{\text{Transfer Function}} U(z)
\end{aligned} \tag{10.20}$$

To correlate the transform-domain solutions in the above expressions with the time-domain expressions in (10.10) and (10.14), it is helpful to note that

$$(zI-A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \cdots \tag{10.21}$$

as may be verified by multiplying both sides by $(zI - A)$. The region of convergence for the series on the right is all values of z outside of some sufficiently large circle in the complex plane. What this series establishes, on comparison with the definition of the $\mathcal{Z}$-transform, is that the inverse transform of $z(zI - A)^{-1}$ is the matrix sequence whose value at time k is A^k for $k \geq 0$; the sequence is 0 for time instants $k < 0$. That is we can write

$$\begin{aligned} \left(I, A, A^2, A^3, A^4, \ldots\right) &\stackrel{\mathcal{Z}}{\longleftrightarrow} z(zI - A)^{-1} \\ \left(0, I, A, A^2, A^3, \ldots\right) &\stackrel{\mathcal{Z}}{\longleftrightarrow} (zI - A)^{-1}. \end{aligned}$$

Also since the inverse transform of a product such as $(zI - A)^{-1}BU(z)$ is the convolution of the sequences whose transforms are $(zI - A)^{-1}B$ and $U(z)$ respectively, we get

$$\begin{aligned} \left(x(0), Ax(0), A^2x(0), A^3x(0), \ldots\right) &\stackrel{\mathcal{Z}}{\longleftrightarrow} z(zI - A)^{-1}x(0) \\ \left(0, B, AB, A^2B, A^3B, \ldots\right) * (u(0), u(1), u(2), u(3), \ldots) &\stackrel{\mathcal{Z}}{\longleftrightarrow} (zI - A)^{-1}BU(z). \end{aligned}$$

Putting the above two pieces together, the parallel between the time-domain expressions and the transform-domain expressions in (10.20) should be clear.

Exercises

Exercise 10.1 **(a)** Give an example of a nonzero matrix whose eigenvalues are all 0.

(b) Show that $A^k = 0$ for some *finite* positive power k if and only if all eigenvalues of A equal 0. Such a matrix is termed *nilpotent*. Argue that $A^n = 0$ for a nilpotent matrix of size n.

(c) If the sizes of the Jordan blocks of the nilpotent matrix A are $n_1 \leq n_2 \leq \ldots \leq n_q$, what is the smallest value of k for which $A^k = 0$?

(d) For an *arbitrary* square matrix A, what is the smallest value of k for which the range of A^{k+1} equals that of A^k? (Hint: Your answer can be stated in terms of the sizes of particular Jordan blocks of A.)

Exercise 10.2 Consider the periodically varying system in Problem 7.4. Find the general form of the solution.

Exercise 10.3 Gambler's Ruin

Consider gambling against a bank of capital A_1 in the following way: a coin is fliped, if the outcome is heads, the bank pays one dollar to the player, and if the outcome is tails, the player payes one dollar to the bank. Suppose the probability of a head is equal to p, the capital of the player is A_2, and the game continues until one party looses all of their capital. Calculate the probability of breaking the bank.

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