# 6.231 DYNAMIC PROGRAMMING

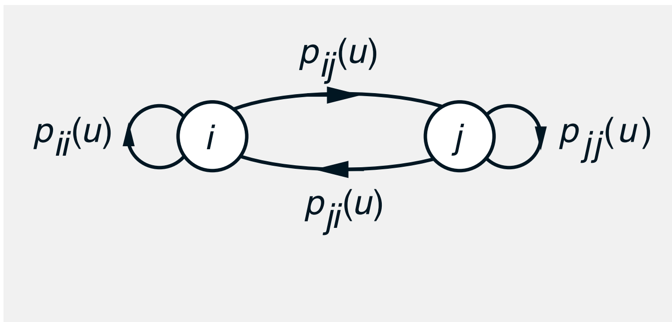
# LECTURE 6

## LECTURE OUTLINE

- Review of Q-factors and Bellman equations for Q-factors
- VI and PI for Q-factors
- Q-learning - Combination of VI and sampling
- Q-learning and cost function approximation
- Approximation in policy space

# DISCOUNTED MDP

- System: Controlled Markov chain with states $i = 1, \ldots, n$ and finite set of controls $u \in U(i)$
- Transition probabilities: $p_{ij}(u)$




- Cost of a policy $\pi = \{\mu_0, \mu_1, \ldots\}$ starting at state $i$:

$$J_\pi(i) = \lim_{N\to\infty} E\left\{\sum_{k=0}^{N} \alpha^k g\big(i_k, \mu_k(i_k), i_{k+1}\big) \mid i = i_0\right\}$$

with $\alpha \in [0, 1)$

- Shorthand notation for DP mappings

$$(TJ)(i) = \min_{u\in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j)+\alpha J(j)\big), \quad i = 1, \ldots, n,$$

$$(T_\mu J)(i) = \sum_{j=1}^{n} p_{ij}\big(\mu(i)\big)\Big(g\big(i,\mu(i),j\big)+\alpha J(j)\Big), \quad i = 1, \ldots, n$$

# THE TWO MAIN ALGORITHMS: VI AND PI

- Value iteration: For any $J \in \Re^n$

$$J^*(i) = \lim_{k\to\infty} (T^k J)(i), \qquad \forall\ i = 1, \ldots, n$$

- Policy iteration: Given $\mu^k$
  - Policy evaluation: Find $J_{\mu^k}$ by solving

$$J_{\mu^k}(i) = \sum_{j=1}^{n} p_{ij}\big(\mu^k(i)\big)\Big(g\big(i, \mu^k(i), j\big) + \alpha J_{\mu^k}(j)\Big), \;\; i = 1, \ldots, n$$

or $J_{\mu^k} = T_{\mu^k} J_{\mu^k}$

  - Policy improvement: Let $\mu^{k+1}$ be such that

$$\mu^{k+1}(i) \in \arg \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i, u, j) + \alpha J_{\mu^k}(j)\big), \quad \forall\ i$$

or $T_{\mu^{k+1}} J_{\mu^k} = T J_{\mu^k}$

- We discussed approximate versions of VI and PI using projection and aggregation

- We focused so far on cost functions and approximation. We now consider Q-factors.

# BELLMAN EQUATIONS FOR $Q$-FACTORS

- The optimal $Q$-factors are defined by

$$Q^*(i,u) = \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J^*(j)\big), \quad \forall\ (i,u)$$

- Since $J^* = TJ^*$, we have $J^*(i) = \min_{u \in U(i)} Q^*(i,u)$ so the optimal $Q$-factors solve the equation

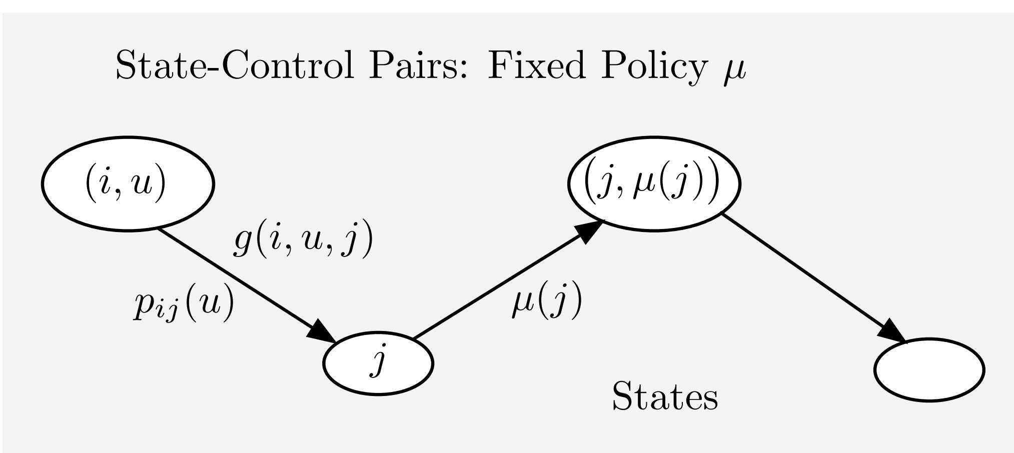
$$Q^*(i,u) = \sum_{j=1}^{n} p_{ij}(u)\left(g(i,u,j) + \alpha \min_{u' \in U(j)} Q^*(j,u')\right)$$

- Equivalently $Q^* = FQ^*$, where

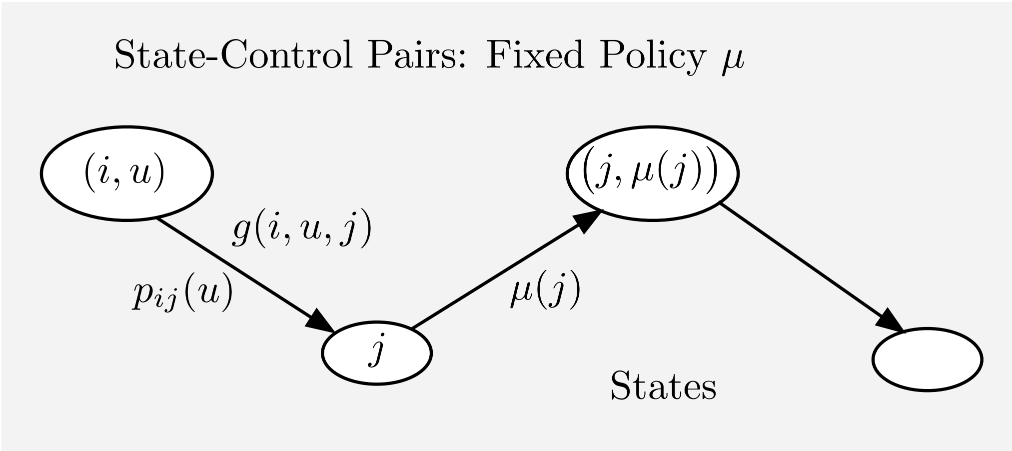
$$(FQ)(i,u) = \sum_{j=1}^{n} p_{ij}(u)\left(g(i,u,j) + \alpha \min_{u' \in U(j)} Q(j,u')\right)$$

- This is Bellman's Eq. for a system whose states are the pairs $(i,u)$

- Similar mapping $F_\mu$ and Bellman equation for a policy $\mu$: $Q_\mu = F_\mu Q_\mu$



## SUMMARY OF BELLMAN EQS FOR $Q$-FACTORS




- Optimal Q-factors: For all $(i, u)$

$$Q^*(i,u) = \sum_{j=1}^{n} p_{ij}(u) \left( g(i,u,j) + \alpha \min_{u' \in U(j)} Q^*(j,u') \right)$$

Equivalently $Q^* = FQ^*$, where

$$(FQ)(i,u) = \sum_{j=1}^{n} p_{ij}(u) \left( g(i,u,j) + \alpha \min_{u' \in U(j)} Q(j,u') \right)$$

- Q-factors of a policy $\mu$: For all $(i, u)$

$$Q_\mu(i,u) = \sum_{j=1}^{n} p_{ij}(u) \Big( g(i,u,j) + \alpha Q_\mu\big(j, \mu(j)\big) \Big)$$

Equivalently $Q_\mu = F_\mu Q_\mu$, where

$$(F_\mu Q)(i,u) = \sum_{j=1}^{n} p_{ij}(u) \Big( g(i,u,j) + \alpha Q\big(j, \mu(j)\big) \Big)$$

# WHAT IS GOOD AND BAD ABOUT Q-FACTORS

- All the exact theory and algorithms for costs applies to Q-factors
  - Bellman's equations, contractions, optimality conditions, convergence of VI and PI
- All the approximate theory and algorithms for costs applies to Q-factors
  - Projected equations, sampling and exploration issues, oscillations, aggregation
- A MODEL-FREE (on-line) controller implementation
  - Once we calculate $Q^*(i,u)$ for all $(i,u)$,

$$\mu^*(i) = \arg\min_{u \in U(i)} Q^*(i,u), \qquad \forall\ i$$

  - Similarly, once we calculate a parametric approximation $\tilde{Q}(i,u,r)$ for all $(i,u)$,

$$\tilde{\mu}(i) = \arg\min_{u \in U(i)} \tilde{Q}(i,u,r), \qquad \forall\ i$$

- The main bad thing: Greater dimension and more storage! [Can be used for large-scale problems only through aggregation, or other cost function approximation.]

# Q-LEARNING

- In addition to the approximate PI methods adapted for Q-factors, there is an important additional algorithm:
  - Q-learning, which can be viewed as a sampled form of VI
- Q-learning algorithm (in its classical form):
  - Sampling: Select sequence of pairs $(i_k, u_k)$ (use any probabilistic mechanism for this, but all pairs $(i, u)$ are chosen infinitely often.)
  - Iteration: For each $k$, select $j_k$ according to $p_{i_k j}(u_k)$. Update just $Q(i_k, u_k)$:

$$Q_{k+1}(i_k,u_k) = (1 - \gamma_k)Q_k(i_k, u_k) + \gamma_k\left(g(i_k, u_k, j_k) + \alpha \min_{u' \in U(j_k)} Q_k(j_k, u')\right)$$

    Leave unchanged all other Q-factors: $Q_{k+1}(i, u) = Q_k(i, u)$ for all $(i, u) \neq (i_k, u_k)$.
  - Stepsize conditions: $\gamma_k$ must converge to 0 at proper rate (e.g., like $1/k$).

# NOTES AND QUESTIONS ABOUT Q-LEARNING

$$Q_{k+1}(i_k,u_k) = (1-\gamma_k)Q_k(i_k,u_k) + \gamma_k\left(g(i_k,u_k,j_k) + \alpha \min_{u'\in U(j_k)} Q_k(j_k,u')\right)$$

- Model free implementation. We just need a simulator that given $(i,u)$ produces next state $j$ and cost $g(i,u,j)$
- Operates on only one state-control pair at a time. Convenient for simulation, no restrictions on sampling method.
- Aims to find the (exactly) optimal Q-factors.
- Why does it converge to $Q^*$?
- Why can't I use a similar algorithm for optimal costs?
- Important mathematical (fine) point: In the $Q$-factor version of Bellman's equation the order of expectation and minimization is reversed relative to the cost version of Bellman's equation:

$$J^*(i) = \min_{u\in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J^*(j)\big)$$

# CONVERGENCE ASPECTS OF Q-LEARNING

- $Q$-learning can be shown to converge to true/exact $Q$-factors (under mild assumptions).
- Proof is sophisticated, based on theories of stochastic approximation and asynchronous algorithms.
- Uses the fact that the Q-learning map $F$:

$$(FQ)(i,u) = E_j\left\{g(i,u,j) + \alpha \min_{u'} Q(j,u')\right\}$$

is a sup-norm contraction.

- Generic stochastic approximation algorithm:
  - Consider generic fixed point problem involving an expectation:

$$x = E_w\{f(x,w)\}$$

  - Assume $E_w\{f(x,w)\}$ is a contraction with respect to some norm, so the iteration

$$x_{k+1} = E_w\{f(x_k,w)\}$$

converges to the unique fixed point

  - Approximate $E_w\{f(x,w)\}$ by sampling

# STOCH. APPROX. CONVERGENCE IDEAS

- For each $k$, obtain samples $\{w_1, \ldots, w_k\}$ and use the approximation

$$x_{k+1} = \frac{1}{k} \sum_{t=1}^{k} f(x_k, w_t) \approx E\big\{f(x_k, w)\big\}$$

- This iteration approximates the convergent fixed point iteration $x_{k+1} = E_w\big\{f(x_k, w)\big\}$
- A major flaw: it requires, for each $k$, the computation of $f(x_k, w_t)$ for all values $w_t$, $t = 1, \ldots, k$.
- This motivates the more convenient iteration

$$x_{k+1} = \frac{1}{k} \sum_{t=1}^{k} f(x_t, w_t), \qquad k = 1, 2, \ldots,$$

that is similar, but requires much less computation; it needs only one value of $f$ per sample $w_t$.

- By denoting $\gamma_k = 1/k$, it can also be written as

$$x_{k+1} = (1 - \gamma_k) x_k + \gamma_k f(x_k, w_k), \quad k = 1, 2, \ldots$$

- Compare with Q-learning, where the fixed point problem is $Q = FQ$

$$(FQ)(i, u) = E_j\Big\{g(i, u, j) + \alpha \min_{u'} Q(j, u')\Big\}$$

# $Q$-FACTOR APROXIMATIONS

- We introduce basis function approximation:

$$\tilde{Q}(i, u, r) = \phi(i, u)' r$$

- We can use approximate policy iteration and LSPE/LSTD for policy evaluation
- Optimistic policy iteration methods are frequently used on a heuristic basis
- Example: Generate trajectory $\{(i_k, u_k) \mid k = 0, 1, \ldots\}$.
- At iteration $k$, given $r_k$ and state/control $(i_k, u_k)$:

(1) Simulate next transition $(i_k, i_{k+1})$ using the transition probabilities $p_{i_k j}(u_k)$.

(2) Generate control $u_{k+1}$ from

$$u_{k+1} = \arg \min_{u \in U(i_{k+1})} \tilde{Q}(i_{k+1}, u, r_k)$$

(3) Update the parameter vector via

$$r_{k+1} = r_k - (\text{LSPE or TD-like correction})$$

- Complex behavior, unclear validity (oscillations, etc). There is solid basis for an important special case: optimal stopping (see text)

# APPROXIMATION IN POLICY SPACE

- We parameterize policies by a vector $r = (r_1, \ldots, r_s)$ (an approximation architecture for policies).

- Each policy $\tilde{\mu}(r) = \{\tilde{\mu}(i; r) \mid i = 1, \ldots, n\}$ defines a cost vector $J_{\tilde{\mu}(r)}$ ( a function of $r$).

- We optimize some measure of $J_{\tilde{\mu}(r)}$ over $r$.

- For example, use a random search, gradient, or other method to minimize over $r$

$$\sum_{i=1}^{n} p_i J_{\tilde{\mu}(r)}(i),$$

where $(p_1, \ldots, p_n)$ is some probability distribution over the states.

- An important special case: Introduce cost approximation architecture $V(i, r)$ that defines indirectly the parameterization of the policies

$$\tilde{\mu}(i; r) = \arg \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u) \big( g(i, u, j) + \alpha V(j, r) \big), \ \forall\, i$$

- Brings in features to approximation in policy space

# APPROXIMATION IN POLICY SPACE METHODS

- Random search methods are straightforward and have scored some impressive successes with challenging problems (e.g., tetris).

- Gradient-type methods (known as policy gradient methods) also have been worked on extensively.

- They move along the gradient with respect to $r$ of

$$\sum_{i=1}^{n} p_i J_{\tilde{\mu}(r)}(i),$$

- There are explicit gradient formulas which have been approximated by simulation

- Policy gradient methods generally suffer by slow convergence, local minima, and excessive simulation noise

# FINAL WORDS AND COMPARISONS

- There is no clear winner among ADP methods
- There is interesting theory in all types of methods (which, however, does not provide ironclad performance guarantees)
- There are major flaws in all methods:
  - Oscillations and exploration issues in approximate PI with projected equations
  - Restrictions on the approximation architecture in approximate PI with aggregation
  - Flakiness of optimization in policy space approximation
- Yet these methods have impressive successes to show with enormously complex problems, for which there is no alternative methodology
- There are also other competing ADP methods (rollout is simple, often successful, and generally reliable; approximate LP is worth considering)
- Theoretical understanding is important and nontrivial
- Practice is an art and a challenge to our creativity!