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# Chapter 7

## Dimensions

The next way to eliminate spurious complexity is the method of dimensional analysis or dimensionless groups.

The following mathematical problem shows how dimensionless groups are, like proportional reasoning and conservation laws, a form of symmetry reasoning. The problem is to expand the polynomial  $(a + b + c)^3$  into its numerous terms. The school-algebra method is to multiply  $a + b + c$  by  $a + b + c$  and then multiply the result by  $a + b + c$ .

The school-algebra method is messy prone to mistakes, but symmetry comes to the rescue. The factor  $a + b + c$  is unchanged if  $a$  and  $b$ , if  $a$  and  $c$ , or if  $b$  and  $c$  are swapped. Indeed, any permutation of  $a$ ,  $b$ , and  $c$  leaves  $a + b + c$  unchanged and leaves  $(a + b + c)^3$  unchanged. Therefore, the product can – and should be – built using invariant combinations of  $a$ ,  $b$ , and  $c$ : combinations that are unchanged by permuting  $a$ ,  $b$ , and  $c$ .

The product contains  $a^3$ , but  $a^3$  alone is not invariant to a permutation. The invariant version of  $a^3$  is  $a^3 + b^3 + c^3$ , which is one component of the product. The product also contains terms like  $a^2b$ , which again is not invariant. The analogous invariant sum comes from adding all possible permutations of  $a^2b$ :

$$(a^2b + ab^2) + (a^2c + ac^2) + (b^2c + bc^2).$$

The third type of term is  $abc$ .

So the product has the form

$$(a + b + c)^3 = A(a^3 + b^3 + c^3) + B(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2) + C(abc),$$

where  $A$ ,  $B$ , and  $C$  are for-the-moment-unknown constants.

Here is one way to evaluate the constants. Set  $a = 1$ ,  $b = c = 0$ . Then the equation reduces to

$$1^3 = A \cdot 1^3,$$

so  $A = 1$ . To get another relation, set  $a = b = c = 1$ . Then, using  $A = 1$ , the equation reduces to

$$27 = 3 + 6B + C.$$

To find  $C$ , notice that there are six ways to get an  $abc$  factor. So  $C = 6$  and then  $B = 3$ .

Thus

$$(a + b + c)^3 = (a^3 + b^3 + c^3) + 3(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2) + 6abc.$$

This symmetry solution has several merits. First, it is less prone to mistakes than is multiplying by brute force. Second, it produces the answer in a meaningful, low-entropy form. The chunks in the solution – the terms  $a^3 + b^3 + c^3$  and  $a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2$  and  $abc$  – each obey the symmetry that nothing important changes if you permute  $a$ ,  $b$ , and  $c$ . Rather than using a brute-force method and then doing hard work to turn the solution into a meaningful form, use symmetry reasoning: Whenever possible, work with quantities that obey the symmetries of the problem.

This chapter shows how this idea leads naturally to dimensionless groups, the fundamental idea of dimensional analysis.

## 7.1 Power of multinationals

The first example shows what happens when people take no notice of dimensions.

Critics of globalization often make this argument:

In Nigeria, a relatively economically strong country, the GDP [gross domestic product] is \$99 billion. The net worth of Exxon is \$119 billion. ‘When multinationals have a net worth higher than the GDP of the country in which they operate, what kind of power relationship are we talking about?’ asks Laura Morosini. [Source: ‘Impunity for Multinationals’, *ATTAC*, 11 Sept 2002, [url:nigeria-argument], retrieved 11 Sept 2006]

Before reading further, try to find the most egregious fault in the comparison between Exxon and Nigeria. It’s a competitive field, but one fault stands out.

The comparison between Exxon and Nigeria has many problems. First, the comparison exaggerates Exxon’s power by using its worldwide assets (net worth) rather than its assets only in Nigeria. On the other hand, Exxon can use its full international power when negotiating with Nigeria, so perhaps the worldwide assets are a fair basis for comparison.

A more serious, and less debatable, problem is the comparison with GDP, or gross domestic product. To see the problem, look at the ingredients in how GDP is usually measured: as dollars per year. The \$99 billion for Nigeria’s GDP is shorthand for \$99 billion per year. A year is an astronomical time, and its use in an economic measurement is arbitrary. Economic flows, which are a social phenomenon, should not care about how long the earth requires to travel around the sun. Suppose instead that the decade was the chosen unit of time in measuring the GDP. Then Nigeria’s GDP would be roughly \$1 trillion per decade (assuming that the \$99 billion per year value held steady) and would be reported as \$1 trillion. Now Nigeria towers over the puny Exxon whose assets are a mere one-tenth of this figure.