6.055J / 2.038J The Art of Approximation in Science and Engineering
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Chapter 4
Symmetry

Symmetry is often thought of as a purely geometric concept, but it is useful in a wide variety of problems. Whenver you can use symmetry, use it and will simplify the solution. The following sections illustrate symmetry in calculus, geometry, and heat transfer.

4.1 Calculus

For what value of x is $3x - x^2$ a maximum?

The usual method is to take the derivative:

$$\frac{d}{dx}(3x - x^2) = 3 - 2x = 0,$$

whereupon $x_{\text{max}} = 3/2$.

Although differentiating is a general method, its generality comes at a cost: that its results are often hard to interpret. One does the manipulations, and whatever formulas show up at the end, so be it. So, if you can find a simplification, you are likely to get a more insight into why the answer came out the way that it did.

For this problem, symmetry simplifies it enough that nothing remains to do. To see how, first factor the equation into $x(3 - x)$. Let x_{max} be where it has its maximum. The factors x and $3 - x$ can be swapped using the substitution $x' = 3 - x$. In terms of x', the problem becomes maximizing $(3 - x')x'$. This formula has the same structure as the original one $x(3 - x)$! So the symmetry operation preserves this structure. Since the x or x' location of the maximum depends only on the structure, the location has the same numerical value whether in the x or x' coordinate systems. So it is said to be invariant under the substitution operation. Therefore, in this problem, the $x' \to 3 - x$ substitution is a symmetry.

Since $x' = 3 - x$ and, as a result of symmetry, $x'_{\text{max}} = x_{\text{max}}$, the only solution is $x_{\text{max}} = x'_{\text{max}} = 3/2$.

A similar, perhaps more telegraphic argument, is that the maximum is halfway between the two roots $x = 0$ and $x = 3$, so the maximum is, again, at $x_{\text{max}} = 3/2$. This argument implicitly contains symmetry, which is the justification for saying that the maximum is midway between the roots.

The next calculus example, from electrical and mechanical engineering, is to maximize the response of a second-order system such as a damped spring–mass system or an *LRC* circuit. The response depends on the frequency and amplitude of the driving input, and is measured as the ratio of output to input amplitude. This ratio is the gain A, and a few applications of Newton's second law produces

$$A(\omega) = \frac{j\omega}{1 + j\omega/Q - \omega^2},$$

where Q is the quality factor of the system (the inverse of the damping), j is $\sqrt{-1}$ and ω is measured in units of the natural frequency.

The problem is to find the peak response, meaning the frequency ω_{max} that maximizes the magnitude of the gain and the gain at that frequency. The magnitude of the gain is

$$|A(\omega)| = \frac{\omega}{\sqrt{(1-\omega^2)^2 + \omega^2/Q^2}}$$

Because of the squares and square roots, a brute-force approach by taking the derivative will generate messy equations. So, use symmetry. What is the symmetry operation? It will be be a flip of the coordinate system, but around what point? The value $\omega = 1$ is special because that choice eliminates the denominator term $(1-\omega^2)^2$, which helps to minimize the denominator and maximize the gain. On the other hand, decreasing ω slightly could increase the gain because, at the cost of increasing $(1-\omega^2)^2$, it decreases the ω^2/Q^2 term in the denominator. On the other hand, increasing ω slightly might produce a higher gain because it increases the numerator of the gain.

To summarize: $\omega = 1$ is special but slightly higher or lower than $\omega = 1$ could be optimal too. Since $\omega = 1$ is special, use it as the point that is preserved by the symmetry operation. For a symmetry operation, interchange the $\omega < 1$ and $\omega > 1$ ranges. Frequencies mostly matter as ratios to one another – for example in music – so do the interchange by defining $\omega' = 1/\omega$ rather than $\omega' = 1 - \omega$. With the reciprocal definition, the problem becomes to maximize the magnitude of $A(\omega')$, where

$$A(\omega') = \frac{j/\omega'}{1 + j/\omega' Q - 1/\omega'^2}.$$

Multiply numerator and denominator by 1 in the form of ω'^2/ω'^2:

$$A(\omega') = \frac{j\omega'}{\omega'^2 + j\omega/Q - 1}.$$

Its magnitude is

$$|A(\omega')| = \frac{\omega'}{\sqrt{(1-\omega'^2)^2 + \omega'^2/Q^2}}.$$

This formula has the same structure as the magnitude in terms of ω itself, and this information is enough to solve for ω_{max}. Because of the isomorphic structure, $\omega'_{\text{max}} = \omega_{\text{max}}$. But by construction $\omega' = 1/\omega$, so ω'_{max} is also $1/\omega_{\text{max}}$. The only solution is $\omega_{\text{max}} = \pm 1$. Since the negative root is boring, the relevant solution is $\omega_{\text{max}} = 1$ and the response there is

$$A(\omega_{\max}) = \frac{j}{1/Q} = jQ.$$

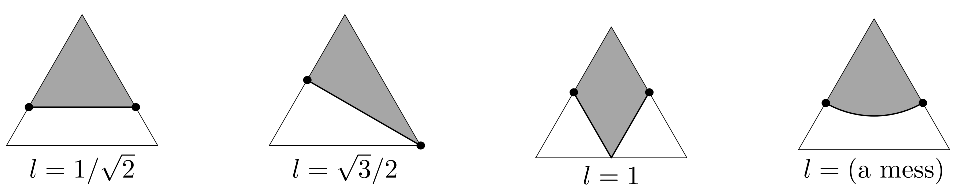
The factor of Q in the maximum response says that a lightly damped system, where $Q \gg 1$, can reach a high amplitude if you push it at the so-called resonant frequency. The j says that the response at this resonant frequency lags the input by 90 degrees. In other words, the greatest push happens when the velocity, not the displacement, is a maximum.

4.2 Graphical symmetry

The following pictorial problem illustrates symmetry applied to a geometric problem, the traditional domain of symmetry:

> How do you cut an equilateral triangle into two equal halves using the shortest, not-necessarily-straight path?

Here are several candidates among the infinite set of possibilities for the path.



Let's compute the lengths of each bisecting path, with length measured in units of the triangle side. The first candidate encloses an equilateral triangle with one-half the area of the original triangle, so the sides of the smaller, shaded triangle are smaller by a factor of $\sqrt{2}$. Thus the path, being one of those sides, has length $1/\sqrt{2}$. In the second choice, the path is an altitude of the original triangle, which means its length is $\sqrt{3}/2$, so it is longer than the first candidate. The third candidate encloses a diamond made from two small equilateral triangles. Each small triangle has one-fourth the area of the original triangle with side length one, so each small triangle has side length 1/2. The bisecting path is two sides of a small triangle, so its length is 1. This candidate is longer than the other two.

The fourth candidate is one-sixth of a circle. To find its length, find the radius r of the circle. One-sixth of the circle has one-half the area of the triangle, so

$$\underbrace{\pi r^2}_{A_{\text{circle}}} = 6 \times \frac{1}{2} A_{\text{triangle}} = 6 \times \frac{1}{2} \times \underbrace{\frac{1}{2} \times 1 \times \frac{\sqrt{3}}{2}}_{A_{\text{triangle}}}.$$

Multiplying the pieces gives

$$\pi r^2 = \frac{3\sqrt{3}}{4},$$

and